



GOVT. POLYTECHNIC BALASORE

[LECTURE NOTES]

**[Signal & Systems]
[TH5]**

**DIPLOMA
IN
3RD SEMESTER, E&TC ENGINEERING**

PREPARED BY:-ER. YOGASAKTI YOGAMAYA

(LECT. IN E & TC)

**DEPT. OF ELECTRONICS & TELECOMMUNICATION
ENGINEERING**

Introduction to Signal and System

Signal And Systems as Seen in everyday life,
and in various branches of engineering & science

- Signal is defined as any physical quantity that varies with time, space or any independent variable.
- A signal can be a function of one or more independent variables such as time, temp, position, pressure or distance.
- If a signal depends on only one variable, it is called one dimensional signal. Ex: Speech signal.
- If a signal depends on two independent variable is called a two dimensional signal. Ex: Picture, X-ray image.
- The speed of wind & air pressure are function of four independent variable: latitude, longitude, elevation & time. These type of signals ^{are} called multidimensional signal.
- Signals can be represented in either time domain or frequency domain.
- The term signal is generally applied to something that conveys information.
- Examples of signal include audio signal (speech, music), image & video signals, communication signals, biomedical signals (ECG, EEG, control systems, digital signal processing, electrical ckts, mechanical systems, and feedback systems).

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dt

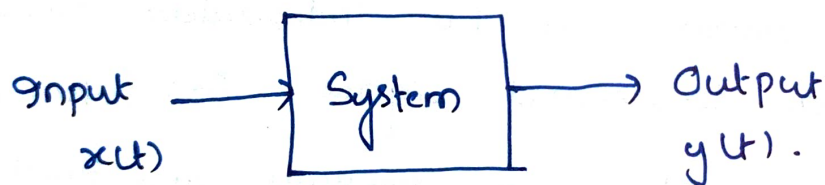
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→ These concepts find practical use in areas such as audio processing, image recognition, telecommunication, medical diagnostics, robotics, power system, & scientific research.

* Signal & Systems provide the framework for analyzing, manipulating and understanding information in various fields.

System:

→ It is a physical device that performs an operation on an input signal & produce another signal as output.



$$y(t) = T[x(t)]$$

- The output $y(t)$ is the transformed form of $x(t)$.
- To study the behaviour of a system is finding a mathematical model of the system.
- A mathematical model consists of a collection of equations describing the relationships between input & output signals of a system.

1.

Electrical, mechanical, hydraulic, thermal, biomedical
signals & system as examples:

1. Electrical Signals & Systems:

- i) ECG - (Electrocardiogram) measures electrical activity of heart.
- ii) EEG (Electroencephalogram) measure electrical activity in brain.
- iii) ENG (Electroneurogram) measure electrical activity in nerves.
- iv) Nervous System: The body's electrical communication network, relaying on electrical signals to transmit information.

2) Mechanical Signals & System:

- i) Force measurement - Signals generated by strain gauges, indicating force applied to an object.
- ii) Displacement measurement - Signals from LVDT or potentiometers indicating linear or angular movement.
- iii) Pendulum - A mechanical system where position & velocity are signals that can be analyzed.

3) Hydraulic Signals & Systems:

- i) Pressure signal - Used to control hydraulic actuators
~~ex: hydraulic pressure~~ ^{ex: hydraulic pressure}.
- ii) Flow Rate of signal :- Indicating the movement of hydraulic fluid.
- iii) Hydraulic Actuators: Systems that use fluid pressure to generate force & motion.

4) Thermal Signals & Systems :

- i) Temperature signals: Measured by devices like resistance thermometer & thermistors.
- ii) Heat flow signals: Indicating the rate of heat transfer.
- iii) Heating & cooling system: Control the temperature based on thermal signals from sensors.

5) Biomedical Signals & System :

- i) Biopotential signals: Includes ECG, EEG & ENG
- ii) Biochemical signals: Includes hormones neuro-transmitters
- iii) Bioacoustic signals: Sound produced by the body
ex: heart sounds.
- iv) Biomedical Instrumentation Systems: Devices designed to measure & analyze physiological signals from the body.
- v) Human Body: A complex system with interacting electrical, chemical, mechanical, thermal & other System.

Extracting the common essence and requirements of signal & system.

* The fundamental essence of signals and systems lies in information and its transformation.

→ Signals are carrier of information while systems process this information, transforming input signals into output signals.

Common Essence :

- i) Information Transfer: Signals are carriers of information.
- ii) System Interaction: Systems are entities that process these signals.
- iii) Understanding Behavior: The field aims to understand the properties of different types of signals & how various systems respond to them.
- iv) Mathematical Representation: Both signals & systems are often described and analyzed using mathematical functions & representation.

Requirements:

- i) Understanding Signal Properties: Familiarity with the characteristics of different signals, such as continuous time vs discrete time, periodic vs aperiodic and deterministic vs random.
- ii) System Properties: Understanding properties like linearity, time invariance, causality & stability.
- iii) Transform Techniques: Utilise different techniques like Fourier Transform, Laplace transform, z-transform etc for analyzing signal & system.
- iv) Convolution & correlation

Classification of signals:

Signals can be classified in following manner.

- 1) classification of signal based on number of dimension
- 2) classification of signal based on number of channels of a signal.
- 3) Based on time
- 4) Based on dependent & independent variable.

1) Classification of signal based on number of dimension:

→ Signals are classified in two groups based on dimension of a signal. The dimension of signal is defined as number of independent variable on which value of signal depends.

- i) One dimensional signal
- ii) Multi dimensional signal.

i) One dimensional signal:

* A signal is called one dimensional signal if it is function of single independent variable.

→ Ex: Speech signal. ~~so~~ Amplitude of signal depends on time.

ii) Multidimensional signal:

* A signal is called multidimensional signal if it is function of more than one independent variable.

→ Ex: Black & white TV signal is two dimensional because brightness of signal depend at each point depend on width & height of picture element (Pixel) i.e. $x \times y$ coordinate.

2) Classification of signal Based on number of channels of a signal

→ Signals are categorized in two groups or based on the number of channels of a signal.

★ The channels of a signal are defined as the number of sources from which it is generated.

They can be

- i) Single channel signal
- ii) Multi channel signal

i) Single channel signal: Signals which are generated by a single source or single sensor are called single channel signals.

Ex: Speech signal. It is generated by a single sensor or by a single source.

ii) Multi channel signal: Signals which are generated by more than one source or sensor are called multichannel signals.

→ These signals are represented by vector.

$$S(t) = \begin{bmatrix} S_1(t) \\ S_2(t) \\ S_3(t) \end{bmatrix}$$

Ex: In electrocardiography 3 lead & 12 lead electrocardiograph is often used which results in 3-channel & 12-channel signals respectively.

3) classification signal based on independent variable i.e Time.

On the basis of condition of Time, signal is classified into two groups.

- i) Continuous Time signal
- ii) Discrete Time signal.

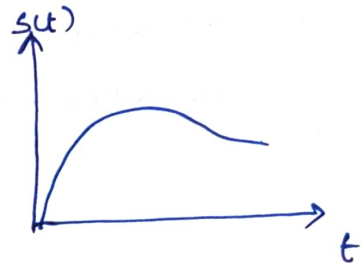
i) Continuous Time Signal:

* A signal that varies continuously with time is called continuous time signal.

→ A continuous-time signal has a value defined at each point in time.

→ A continuous time signal has a value is denoted by a function of time such as $S(t)$ where t is continuous time independent variable.

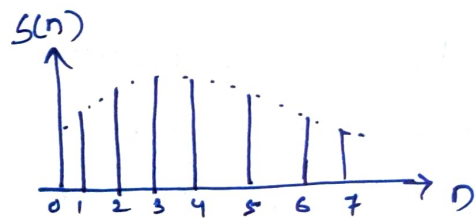
Ex: Speech signal, temp. of Room.



ii) Discrete-time signal:

* Discrete-time signals are signals which are defined at discrete time.

→ These are represented by sequences of numbers.



→ A discrete-time signal has a value defined only at discrete points in time.

→ A discrete-time signal is denoted by $S(n)$ where n is discrete-time independent variable.

→ Ex: Year wise Population, no. of vehicle transmit in every hour.

4) classification of signal based on condition of Dependent variable & Independent variable of a signal :

On this basis, signals are classified in two groups.

i) Analog signal.

ii) Digital signal.

i) Analog Signals: Analog signals are signal whose both dependent & independent variable are continuous in nature.

- Analog signal arise when a physical waveform is converted into an electrical signal.

Ex: Telephone speech signal, TV signal etc.

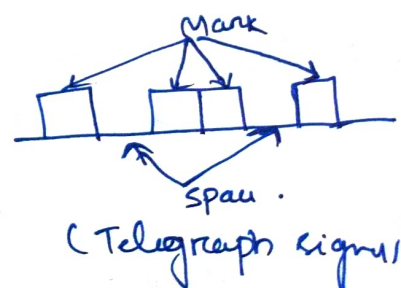
- Continuous amplitude, continuous ~~amplitude~~ time signals are called analog signal.



ii) Digital Signals: Digital signals are signal whose both dependent variable & independent variable are discrete in nature.

→ Digital signals comprise pulse occurring at discrete intervals of time.

Ex: Telegraph & Teleprinter signals are examples of digital signal.



(Telegraph signal)

General Signal characteristics: / Classification of Continuous Time & Discrete Time signals:

Continuous-time & discrete Time signals further classified in terms of various characteristic they possess.

1. Deterministic & random signals
2. Periodic & ~~non~~ Aperiodic signals
3. Even & odd signals
4. Energy & Power signal.
5. Causal & Non causal signal

i) Energy & Power signal:

→ A signal is called Energy signal if the energy of the signal is finite & ~~zero~~ average power is zero.
 $E = \text{finite}, P = 0$

→ A signal is called power signal if the power is finite & energy is infinite.
 $P = \text{finite}, E = \infty$

→ ~~Signal~~

→ A signal is neither an energy signal nor a power signal if it has infinite energy & zero power.

Ex: Ramp signal

Instantaneous power across a resistor is

$$P(t) = v(t) i(t) \quad \text{--- (1)}$$

$$\text{But } i(t) = \frac{v(t)}{R}$$

By substituting in eqⁿ (1)

$$P(t) = v(t) \cdot \frac{v(t)}{R} = \frac{v^2(t)}{R}$$

Hence total energy at R equals

$$E = \int_{-\infty}^{\infty} p(t) dt = \int_{-\infty}^{\infty} \frac{v^2(t)}{R} dt$$
$$= \frac{1}{R} \int_{-\infty}^{\infty} v^2(t) dt$$

if $R=1/2$ then $E = \int_{-\infty}^{\infty} v^2(t) dt$.

Hence For any continuous signal $x(t)$, the energy E can be written as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Where $|x(t)|$ denotes the magnitude of the signal $x(t)$

Similarly the total energy of in a discrete time signal $x(n)$ can be written as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The average power of a periodic ~~cont~~ continuous signal $x(t)$ which is ~~periodic~~ periodic with period T can be written as

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

The power of a discrete time signal $x(n)$ can be written as

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2$$

Ex Determine the energy & power of signal .

1) $x(t) = 2e^{-at} u(t)$ a70

2) $x(t) = t^{1/4} u(t-2)$

3) $x(n) = 3(0.5)^n u(n)$

Ans/1. Given that $x(t) = 2e^{-at} u(t)$ a70

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= \int_{-\infty}^{\infty} |2e^{-at} u(t)|^2 dt \\ &= \int_0^{\infty} |2e^{-at} \cdot 1|^2 dt \\ &= \int_0^{\infty} 4e^{-2at} dt \\ &= 4 \int_0^{\infty} e^{-2at} dt \\ &= 4 \left[\frac{e^{-2at}}{-2a} \right]_0^{\infty} \\ &= \frac{4}{-2a} [e^{-\infty} - e^0] \\ &= \frac{4}{-2a} [0 - 1] = \frac{-4}{-2a} = \frac{2}{a} \end{aligned}$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |x(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |2e^{-at}|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T 4e^{-2at} dt$$

$$= \lim_{T \rightarrow \infty} \frac{4}{T} \int_0^T e^{-2at} dt$$

$$= \lim_{T \rightarrow \infty} \frac{4}{T} \left[\frac{e^{-2at}}{-2a} \right]_0^T$$

$$= \lim_{T \rightarrow \infty} \frac{4}{-2aT} [e^{-2aT} - e^{-0}]$$

$$= 0$$

Since $P = 0$ & $E = \text{finite}$, it is a ~~an~~ energy signal

$$2) x(t) = t^{-1/4} u(t-2)$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |t^{-1/4} u(t-2)|^2 dt$$

$$= \int_2^{\infty} t^{-2/4} dt = \int_2^{\infty} t^{-1/2} dt = \left[\frac{t^{1/2}}{1/2} \right]_2^{\infty} = \infty$$

Power of the signal.

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |x(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |t^{-1/4} u(t-2)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_2^T t^{-2/4} dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_2^T t^{-1/2} dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \left[\frac{t^{1/2}}{1/2} \right]_2^T = \lim_{T \rightarrow \infty} \frac{2}{T} [t^{1/2}]_2^T$$

$$= \lim_{T \rightarrow \infty} \frac{2}{T} [T^{1/2} - 2^{1/2}] = \lim_{T \rightarrow \infty} \frac{2}{T} [\sqrt{T} - \sqrt{2}]$$

$$\Rightarrow P = 0$$

Since $P = 0$, and $E = \infty$, it is neither an energy signal nor a power signal.

3) Given $x(n) = 3(0.5)^n u(n)$

Energy of the signal

$$\begin{aligned} E &= \sum_{n=-\infty}^{\infty} |x(n)|^2 \\ &= \sum_{n=0}^{\infty} |3(0.5)^n u(n)|^2 \\ &= \sum_{n=0}^{\infty} 9(0.5)^{2n} \\ &= 9 \sum_{n=0}^{\infty} (0.25)^n \end{aligned}$$

using $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ for $|a| < 1$

$$\Rightarrow E = 9 \left(\frac{1}{1-0.25} \right)$$

$$= 9 \left(\frac{1}{0.75} \right) = 9 \times 1.333 = 12$$

Power of the signal

$$P = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |3(0.5)^n|^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} 9(0.25)^n$$

Using $\sum_{n=0}^{N-1} a^n = \frac{1-a^{N+1}}{1-a}$

$$a \neq 1$$

$$\int_0^{\infty} \omega t$$

$$P = \lim_{N \rightarrow \infty} \frac{9}{N} \left[\frac{1 - (0.25)^N}{1 - 0.25} \right]$$

$$= 0$$

As $E = \text{finite}$ & $P = 0$ It is an energy signal.

Signal Properties:

i) Periodicity:

Periodicity is the property of the signal i.e. the signal is repeating itself after specific time period T . Such that

$$\boxed{x(t+T) = x(t)} \quad -\infty \leq t \leq \infty \quad \text{--- (1)}$$

Here T is the smallest positive value for which eqn (1) holds is called fundamental period of signal.

A regular example of periodic signal is sinusoidal signal

$$x(t) = A \sin(\omega_0 t + \phi) = A \sin(2\pi F_0 t + \phi)$$

where A represents amplitude, ω_0 is frequency in rad/sec, ϕ is the phase in radians & F_0 frequency is

H2



→ The fundamental period T for this signal can be

$$\boxed{T = \frac{2\pi}{\omega_0} = \frac{1}{F_0}}$$

→ The sum of two periodic signal is periodic only if the ratio of their respective periods can be expressed as a rational number.

For discrete signal $x(n+N) = x(n)$ for all n .

Where N is positive integer & period of discrete time signal. It means that the signal $x(n)$ repeats itself every N samples.



→ The smallest N for which above condition holds is called Fundamental Period N_0 .

ii) Absolute Integrability

* Absolute integrability is a property that

describes the integrability of a signal over its defined domain.

→ A signal is said to be absolutely integrable if the integral of its absolute value exists and is finite.

→ Mathematically, if the integral of $|f(t)|$ from $-\infty$ to ∞ is finite, then the signal is absolutely integrable.

→ This property is important in signal processing and analysis as it ensures that the energy ~~the~~ or power of the signal is well-defined.

(iii) Determinism:

* Determinism ~~also~~ refers to the ~~any~~ predictable property of a signal that follows a ~~predictable~~ and deterministic pattern.

* A signal is said to be ~~deterministic~~ ^{deterministic} if there is no uncertainty ~~in~~ w.r.t its value at any instant of time on signal which can be

defined exactly by a mathematical ~~functions~~ ~~at~~
formula are known as deterministic signal.

Ex: Deterministic signals
→ ~~include~~ include most
mathematical functions ~~such~~ as
~~Polynomial~~ ~~Polynomial~~ ~~functions~~ functions
or ~~exponential~~ exponential functions.



(ii) ~~Stochastic~~ stochastic character:

→ Stochastic character, also known as randomness or probabilistic behavior refers to the property of a signal that ~~exhibits~~ exhibits randomness or uncertainty.

→ Stochastic signals are characterized by having a random or unpredictable nature, & their future value cannot be precisely determined.

→ Stochastic signals are ~~of~~ often modelled using statistical techniques and probability distributions.

→ Example of stochastic signal include noise signal, stock market fluctuation or ~~the~~ weather patterns.

Basic Signals

There are many elementary signals which must be required to better understand the properties of both signal & system. These are includes

- i) Unit impulse
- ii) Unit step
- iii) The Sincusoid
- iv) The complex exponential
- v) Ramp
- vi) Signum

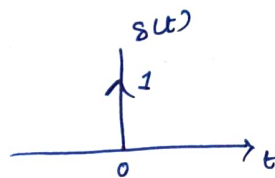
i) The Unit impulse:

Continuous-Time Unit impulse:

The unit impulse denoted as $\delta(t)$ is also known as ~~Dirac~~ Dirac delta function.

This function is defined as

$$\delta(t) = \begin{cases} 1 & \text{for } t=0 \\ 0 & \text{for } t \neq 0 \end{cases}$$



The arrow on the top of the impulse indicates that it has infinite amplitude.

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

- i) The width of pulse is zero which means the pulse exists only at origin ($t=0$).
- ii) The height of the pulse is infinite.
- iii) The area under the curve is unit.
- iv) The height of the arrow represents total area under the pulse curve.

Properties of Unit impulse:

$$1. \int_{-\infty}^{\infty} x(t) \cdot \delta(t) dt = x(0)$$

1. $\int_{-\infty}^{\infty} x(t) \cdot \delta(t) dt = x(0)$ (Sampling Property) .
i.e if we multiply the impulse function by any signal $x(t)$ & integrate it, it picks the value of the function $x(t)$ at $t=0$.

2. Shifting Property: $\int_{-\infty}^{\infty} x(t) \cdot \delta(t-t_0) dt = x(t_0)$

3. Scaling Property: $\delta(at) = \frac{1}{|a|} \delta(t)$ $a \neq 0$

4. Product Property = $x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$
Similarly $x(t) \cdot \delta(t-t_0) = x(t_0) \cdot \delta(t-t_0)$

5. Time Reversal Property: $\delta(-t) = \delta(t)$

NOTE : Importance of Impulse Function

→ By applying impulse signal to a system one can get the impulse response of the system.

→ From impulse response, it is possible to get the transfer function of the system.

→ From the impulse response of the system, one can easily get the step response & ramp response by integrating it once ~~two~~ and twice respectively.

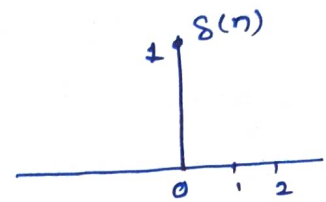
→ For a linear time invariant system, if the area under the impulse curve is finite, then the system is said to be stable.

- Impulse ~~response~~ signal is easy to generate & apply to any system.

Discrete time Unit impulse Function / Sequence:

- It is the simplest basic discrete time signal.
- It is represented by $\delta(n)$ & is defined by

$$\delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{otherwise} \end{cases}$$



Graphically $\delta(n)$ is represented as

2) The ~~impulse~~ Unit Step

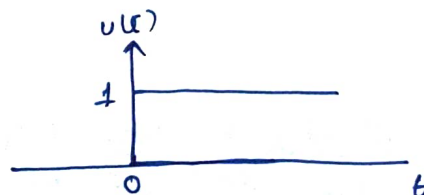
a) Continuous Time Unit Step Function:

The unit step signal is denoted by $u(t)$.

Mathematically it is defined as

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Graphically



- Unit-step function $u(t)$ is discontinuous at time $t=0$.

Relationship betⁿ Unit impulse & Unit step:

The unit step function ~~u(t)~~ $u(t)$ is the running integral of the unit-impulse function $\delta(t)$.

$$u(t) = \int_{-\infty}^t \delta(z) \cdot dz$$

We can say unit impulse function $\delta(t)$ is the first derivative of unit step function $u(t)$.

$$\delta(t) = \frac{du(t)}{dt}$$

Since $u(t)$ is discontinuous at $t=0$, it is not formally differentiable. but above eqn $(\delta u) = \frac{d}{dt} u(t)$ can be interpreted as approximation to unit step function.

Importance of Step function:

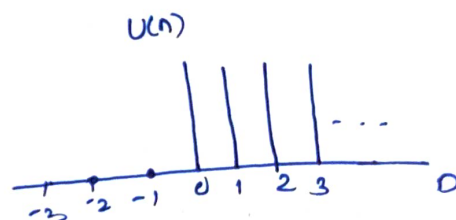
- Unit Step signal is easy to generate & apply to the system, is used in mathematics of control system & signal processing which switches on at a specified time & stays on indefinitely.

DT Unit step Sequence is:

A discrete time unit step sequence can be represented by

$$U(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Graphical Representation:



Relationship betⁿ Unit Step & Unit impulse

→ unit impulse $\delta(n)$ is the first difference betⁿ ~~unit~~ unit step sequence, $u(n)$. $\delta(n) = U(n) - U(n-1)$

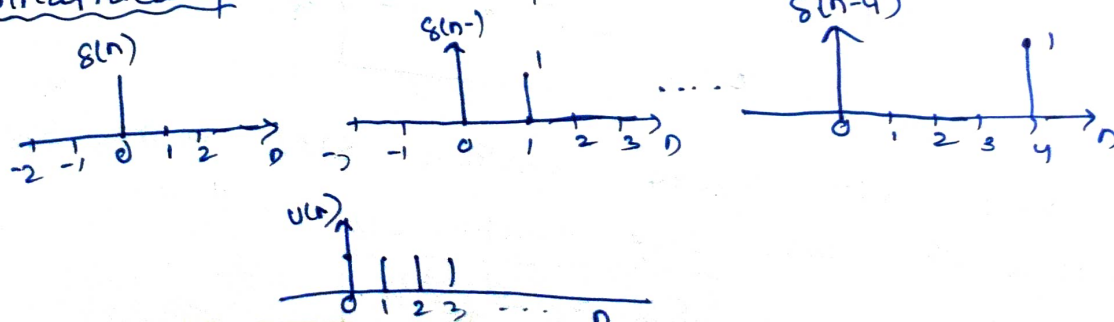
→ The unit step sequence is the running sum of the unit-sample sequence $\delta(n)$.

$$U(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

$$U(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

$$\Rightarrow U(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \dots$$

Graphically



Ex: $x(n) = \{2, -4, 8, 2, 6, 2\}$. Represent the ~~signal~~ $x(n)$ in terms of weighted shifted impulse functions.

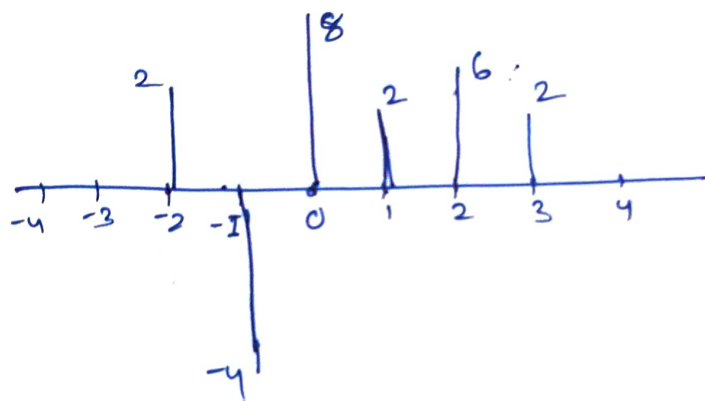
Ans:

Here range of $x(n)$ is -2 to 3

$$\text{So } x(n) = \sum_{k=-2}^3 x(k) \cdot \delta(n-k)$$

By expanding equation

$$x(n) = x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + x(3)\delta(n-3)$$



iii) The Sinusoid:

i) CT Sinusoid signal:

Sinusoidal signal is given by

$$s(t) = A \cos(\omega_0 t + \phi_0)$$

where A = Amplitude of signal

t = continuous time

ω_0 = angular fundamental frequency

ϕ_0 = Phase angle in radian.

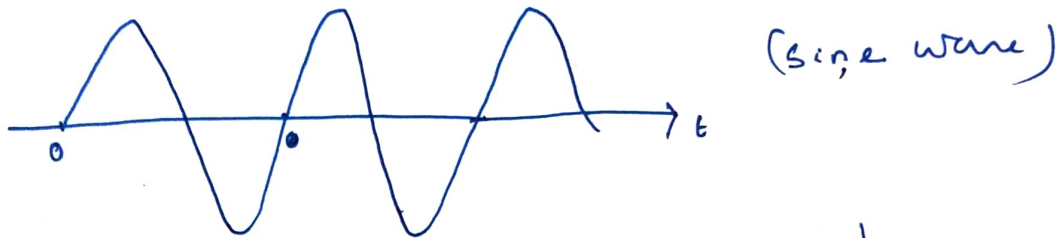
$$\omega_0 = 2\pi f_0$$

f_0 = fundamental frequency in Hz.

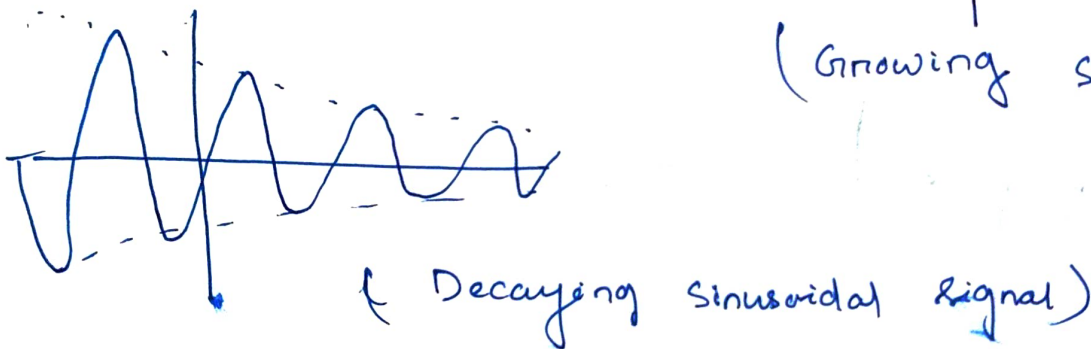
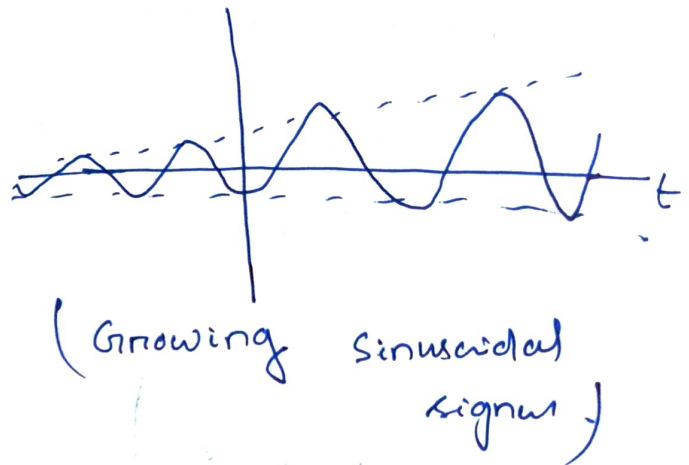
Sinusoidal signal is also periodic with fundamental period T_0 .

- Sinusoidal signal can be represented in terms of periodic complex exponential signal

$$s(t) = A [\cos(\omega_0 t + \phi) + j \sin(\omega_0 t + \phi)]$$



DT Sinusoidal Sequence:



DT Sinusoidal Sequence:

It is represented by real & imaginary parts

$$x(n) = A [\cos(\omega_0 n + \phi) + j \sin(\omega_0 n + \phi)]$$

when $\cos(\omega_0 n + \phi)$ & $\sin(\omega_0 n + \phi)$ are discrete time sinusoidal sequence.

(4) Exponential Signal:

i) CT Exponential Signal:

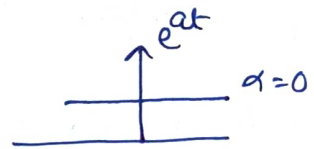
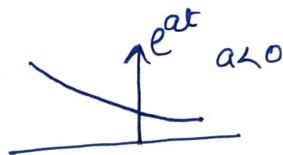
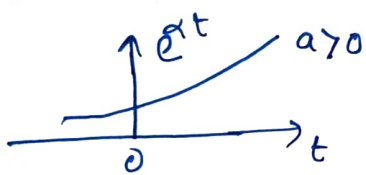
It is represented by $x(t) = Ae^{at}$

Depending upon the value of parameters A & a , the exponential signal is categorized in two types.

- a) Real Exponential Signal
- b) Complex Exponential Signal.

a) Real Exponential Signal:

For this signal A & a are real. Depending upon the value of a , the signal increases or decreases exponentially.



b) Complex Exponential Signal:

For exponential signal $x(t) = Ae^{at}$ where A & a are complex value.

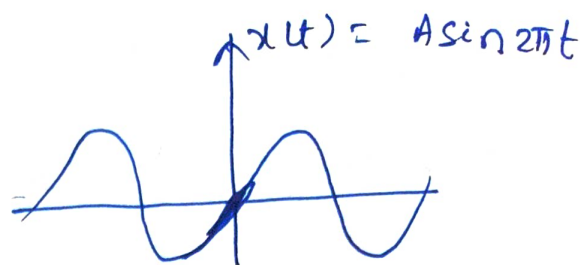
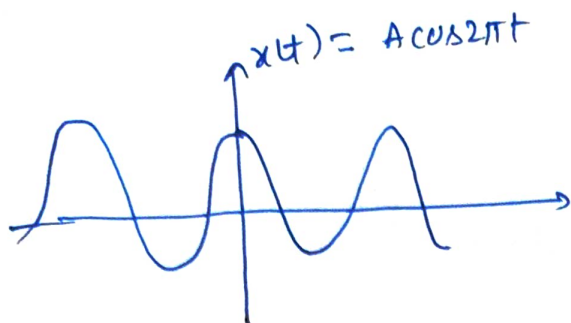
where $x(t)$ can be written as

$$\begin{aligned} x(t) &= |A| e^{j\phi} e^{(\sigma + j\omega_0)t} \\ &= |A| e^{\sigma t} e^{j(\omega_0 t + \phi)} \end{aligned}$$

$$x(t) = |A| e^{\sigma t} [\cos(\omega_0 t + \phi) + j \sin(\omega_0 t + \phi)] + j$$

The complex exponential signal can be represented graphically in terms of sine & cosine waves depending on the value of σ in eqn (1).

→ If the value of $\sigma = 0$, real & imaginary part of eqn (1) becomes sinusoidal signal as shown.



(CT complex exponential signal)

(11) DT Exponential signal:

It is defined as $x(n) = A d^n$

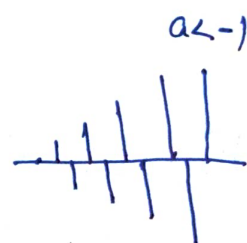
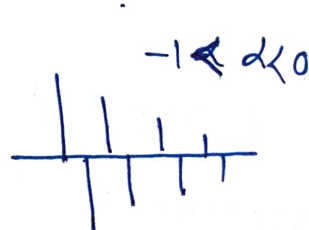
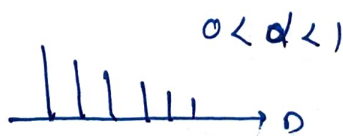
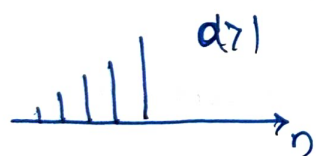
Depending upon values of parameter A & d , DT exponential sequences are 2 types -

(a) Real Exponential Sequence.

(b) Complex Exponential Sequence.

(a) Real Exponential Sequence:

For this sequence A & d are real.



b) Complex Exponential Sequence:

Here $x(n)$ can be written as

$$x(n) = |A| e^{j\theta} |d|^n e^{j(\omega n + \phi)}$$

$$x(n) = |A| |d|^n e^{j(\omega n + \theta)}$$

$$\Rightarrow x(n) = |A| |d|^n [\cos(\omega n + \theta) + j \sin(\omega n + \theta)]$$

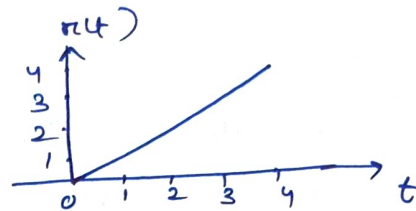
(5) Ramp Sequence:

i) CT Ramp Sequence:

It is represented by $r(t)$. It is defined as

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Graphical Representation



$r(t)$ can be expressed in terms of $u(t)$ as

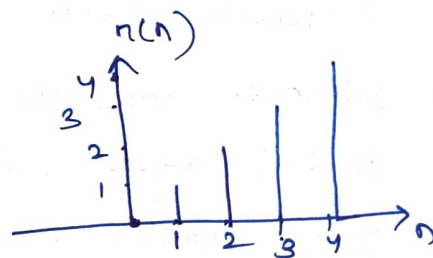
$$r(t) = t u(t)$$

ii) DT Ramp Sequence:

A DT ramp sequence is represented by $r(n)$ is given by

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & n < 0 \end{cases}$$

Graphical Representation



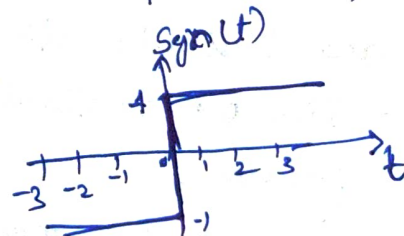
(6) Signum Function

i) CT Signum Function:

The signum function is denoted as $\text{sgn}(t)$ & represented by

$$\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

Graphically

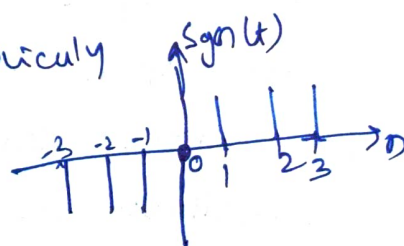


ii) DT Signum Function:

It can be written as

$$\text{sgn}(n) = \begin{cases} 1 & n > 0 \\ 0 & n = 0 \\ -1 & n < 0 \end{cases}$$

Graphically



In terms of unit step, $\text{sgn}(n) = u(n) - u(-n)$

Some Special time-limited Signals:

i) Continuous Amplitude Signals:

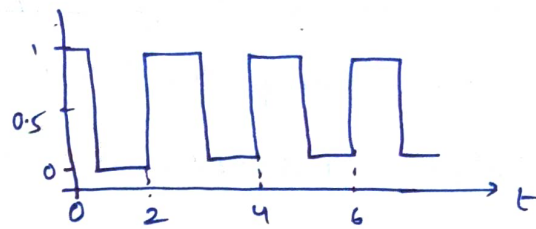
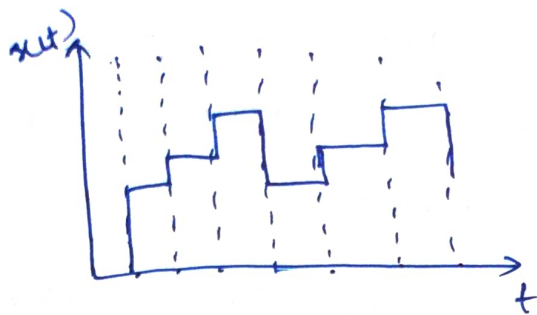
- Continuous amplitude signals can take ^{on} any value within a continuous range.
- The amplitude of the signals can vary continuously over time.
- These signals are often encountered in analog system, where the signal can have an infinite number of possible amplitude value.
- Analog audio signals & continuous waveform signals are examples of continuous amplitude signals.

ii) Discrete Amplitude Signals:

- Discrete Amplitude signals can only take ~~an~~ on a finite set of values. The amplitude of the signal is restricted to specific levels or values.
- Discrete amplitude signals are commonly encountered in digital system, where amplitude is represented using a finite number of bits.
- Digital audio signals & discrete waveform signals are examples of discrete amplitude signals.

* Continuous time discrete amplitude signals are basically digital signals.

- Discrete amplitude signal can be found by quantization process.
- Quantization means assigning the amplitude value of any analog signal to certain discrete levels, equidistant of each other based on certain criteria.
- A square wave is ~~discrete-time~~ a continuous time discrete amplitude signal.



System: A system can be recognized as an interconnection of physical components or subsystems or software to perform the activities or tasks.

→ It operates on the provided input signal, processing it accordingly & provides output.

System Properties:

i) Linearity: The property of the system that obey superposition principle is called linearity & that system is called Linear system. [Superposition Principle stated as response of the system to weighted sum of input signals be equal to corresponding weighted sum of response of the system to each of the individual input signal.

Mathematically: In a CT System if input $x_1(t)$ produce o/p $y_1(t)$ & input $x_2(t)$ produce output $y_2(t)$ then

$$a_1 x_1(t) + a_2 x_2(t) \rightarrow a_1 y_1(t) + a_2 y_2(t)$$

For a DT system if input $x_1(n)$ produce output $y_1(n)$ & input $x_2(n)$ produce output $y_2(n)$ then

$$a_1 x_1(n) + a_2 x_2(n) \rightarrow a_1 y_1(n) + a_2 y_2(n)$$

ii) Additivity Property: Additivity means that the response of the system to the sum of two input equals to the sum of individual response to each point.

Mathematically: If input $x_1(t)$ produces output $y_1(t)$ & input $x_2(t)$ produces output $y_2(t)$ then

$$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$$

iii) Homogeneity / Scaling Property: Homogeneity means that the scaling the input signal by a constant scales the output response by ^{the} same constant.

Mathematically: If input $x(t)$ is scaled by a constant a i.e. $a x(t)$ then it must produce scaled output $a y(t)$.

iv) Realizability / Realizability:

* Realizability is a property that determines whether a given system can be implemented physically.

→ It considers practical considerations like available resources, physical constraints & feasibility.

- A realizable system can be physically built & operated.

Ex: Consider a system $y(t) = x(t-1)$ is causal system because its output is time-delayed version of the original signal.

- Consider a 2nd system $y(t) = x(t+1)$ is non-causal because its output is time advanced of the input signal. This means at a ^{output} time $t=0$, the system requires access to the value of the input signal at time $t=1$. So it is impossible in a realizable system as no body can predict future.

✓ Shift invariance / Time invariant:

→ A System is called as time invariant if its behaviour does not change w.r.t time.

→ That means if a system is provided an shifted input in time ' t_0 ', then the system will produce an output shifted by same time ' t_0 '.

Ex: If input $x(t)$ is delay by t_0 i.e. $x(t-t_0)$ then system produce delayed output $y(t-t_0)$.

Procedure:

- Delay the input signal by ' t_0 ' & check output of system $y_1(t)$
- Delay the output of the system for unshifted input by ' t_0 '. Let this delayed output $y_2(t)$.

~~check $y_1(t) = y_2(t)$~~

- check whether $y_1(t) = y_2(t)$. If they are equal, system is time invariant.

Problem : check following systems time invariant or not.

1. $y(t) = 5t x(t)$
2. $y(t) = x(t) \sin(10\pi t)$
3. $y(t) = 3x(t^2)$
4. $y(t) = 4e^{x(t)}$
5. $y(t) = t^2$

Ans.

1. $y(t) = 5t x(t)$

Delay input by t_0 ,

Delay output by t_0 ,

$$y_1(t) = 5t x(t-t_0)$$

$$y_2(t) = 5(t-t_0) x(t-t_0)$$

As $y_1(t) \neq y_2(t)$
System is not time invariant.

$$2. y(t) = x(t) \sin(10\pi t)$$

$$\text{Delay input by } t_0, \quad y_1(t) = x(t-t_0) \sin(10\pi t)$$

$$\text{Delay output by } t_0, \quad y_2(t) = x(t-t_0) \sin[10\pi(t-t_0)]$$

Hence $y_1 \neq y_2$, hence system is time variant.

$$3. y(t) = 3x(t^2)$$

$$\text{Delay input by } t_0, \quad y_1(t) = 3x(t^2 - t_0)$$

$$\text{Delay output by } t_0, \quad y_2(t) = 3x(t-t_0)^2$$

Hence $y_1 \neq y_2$, hence system is time variant.

$$4. \text{ ~~g(t)~~ } y(t) = 4e^{x(t)}$$

$$\text{input delay by } t_0, \quad y_1 = 4e^{\text{~~x(t-t_0)~~}}$$

$$4. y(t) = 4e^{x(t)}$$

$$\text{input delayed by } t_0, \quad y_1(t) = 4e^{x(t-t_0)}$$

$$\text{Output delayed by } t_0, \quad y_2(t) = 4e^{x(t-t_0)}$$

As $y_1(t) = y_2(t)$ System is time invariant.

$$5. y(t) = t^2$$

$$\text{i/p delay by } t_0, \quad y_1(t) = (t-t_0)^2$$

$$\text{o/p " " } y_2(t) = (t-t_0)^2$$

$y_1(t) \neq y_2(t)$ System time variant.

v) Causality: It is the property of the system which produce output response that depends only on present & past value of the signal.

→ Causality means that the output of the system does not depend on future input but only on past input.

→ The output of a non causal system depends upon present, past as well as future value of input signals.

Problem: Determine systems are causal or noncausal.

1. $y(t) = x(t+1)$
2. $y(t) = x(t) + x(t-1)$
3. $y(n) = nx(n) + x(n-3)$

Ans:

1. $y(t) = x(t+1)$

for $y(0) = x(1)$

for $y(1) = x(2)$

Here output depends future value of input. Hence it's a noncausal system.

2. $y(t) = x(t) + x(t-1)$

$y(0) = x(0) + x(-1)$

$y(1) = x(1) + x(0)$

Here output depends on present as well as past value of input signal. So it is a causal system.

3. $y(n) = nx(n) + x(n-3)$

$y(0) = 0 \cdot x(0) + x(-3)$

$y(1) = 1 \cdot x(1) + x(-2)$

Output depends on present & past value of input. Hence it is a causal ~~system~~ system.

VII) Stability:

→ Stability refers to boundedness of the system response. A system is considered stable if for the bounded input signal, the output remains bounded. i.e. small input lead to output that do not diverge.

→ Ex: if we apply only little pressure to push the object, it will move only a little bit.

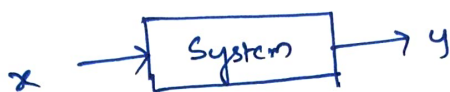
→ In other words if the input signal is finite, the output signal should also be finite.

Linear Shift - invariant (LSI) System in details :

* A System which possesses two properties i.e. linearity and time invariance is called linear shift/time invariance (LSI/LTI) System.

→ A System is called Linear System if the system follows the linear scaling & superposition principle.

Linear scaling:

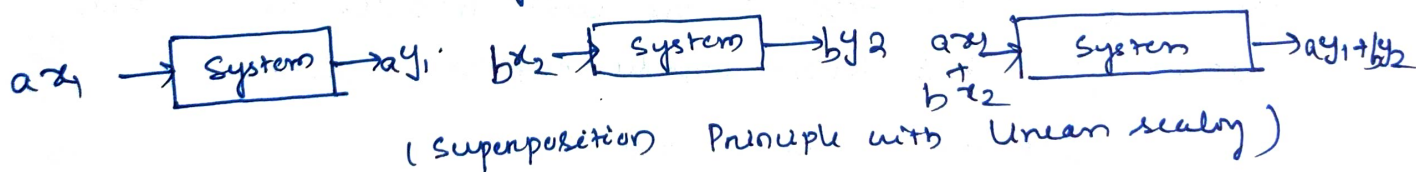


then



When the input to a given system is scaled by a constant value, if the output of the system is also scaled by same amount then the system is Linear scaling.

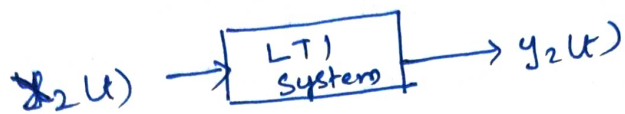
Superposition Principle: If two inputs are added together & passed through a linear system, the output will be sum of the outputs corresponding to their individual inputs.



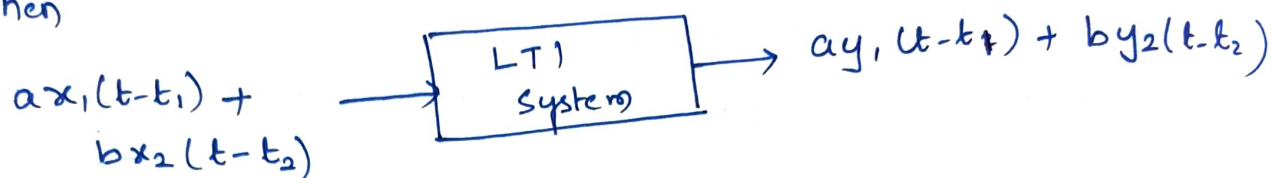
Time invariant System: It has the property that a certain input will always give the same output without regard to when the input was applied to the system.



Hence the linear time/shift invariant system

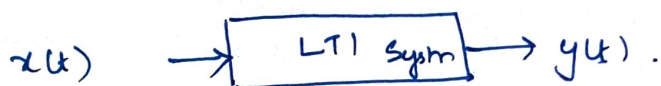


then



Impulse response of LTI system:

a. Continuous Time:



The output of LTI system is called unit-impulse response when we apply an input equal to unit impulse function.

i.e

If $x(t) = \delta(t)$ ~~this~~ is the unit impulse function then output $y(t) = h(t)$ is unit impulse response where

$$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

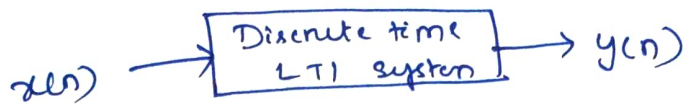
Transfer function of a ~~continuous~~ LTI system is equal to Laplace transform of unit impulse response of ~~the~~ LTI system.

$$\boxed{H(s) = \text{Laplace transform of } h(t)}$$

$$H(s) = \boxed{\text{LT } [h(t)] = \int_{-\infty}^{\infty} h(t) e^{-st} dt}$$

Where s is the complex variable & $s = \sigma + j\omega$, σ is real & ω is imaginary part.

Discrete time ~~discrete time~~



Output of a discrete time LTI system is called unit impulse response or unit sample response when we apply unit impulse sequence $\delta(n)$ as input.

If input $x(n) = \delta(n)$ unit impulse sequence or unit sample sequence, $y(n) = h(n)$ unit impulse response where

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

Transfer function of a discrete time LTI system is equal to Z transform of unit sample response of discrete time LTI system.

$$H(z) = Z[h(n)] = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

When z is complex variable $z = re^{j\omega}$, r is the magnitude of z & ω is the angle of z .

*

Convolution Integral for Continuous LTI system:

* Convolution is a mathematical operation applied to two functions of the same independent variable & result is a third function of the same independent variable. The third function is called the convolution of two functions.

→ Convolution of two continuous variable functions is called continuous convolution. It is also called "convolution integral".

→ Convolution integral is referred to as superposition integral. It is used to compute the response of a continuous-time LTI system with impulse response $h(t)$ & for any arbitrary input $x(t)$.

→ Impulse response $h(t)$ is the output of system when unit impulse function $\delta(t)$ is applied as input.

→ Any input signal $x(t)$ can be represented in terms of unit-impulse function $\delta(t)$ given by

$$x(t) = \int_{-\infty}^{\infty} x(z) \cdot \delta(t-z) dz$$

When $\delta(t) =$ Unit impulse function $= \begin{cases} 1, & t=0 \\ 0, & t \neq 0 \end{cases}$

Convolution integral is given as

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(z) h(t-z) dz$$

Here convolution is denoted by $*$.

Convolution Sum of Discrete-Time LTI System:

→ Convolution of two discrete-variable (i.e. time) functions is called discrete convolution.

→ It is also called "convolution sum."

→ It is used to compute response of a discrete-time LTI system with impulse response $h(n)$ for any arbitrary sequence $x(n)$.

~~$x(t)$~~

→ Any input $x(n)$ can be represented in terms of its unit-impulse sequence $\delta(n)$ which can be represented

by

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n-k)$$

The convolution sum is also called Superposition sum & it is given as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

→ Any input signal $x(t)$ can be represented in terms of unit-impulse function $\delta(t)$ given by

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau) d\tau$$

where $\delta(t) =$ unit impulse function $\begin{cases} 1 & t=0 \\ 0 & t \neq 0 \end{cases}$

convolution integral is given as

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

Here convolution is denoted by $*$.

Convolution sum of Discrete-Time LTI System:

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→ It is also called "convolution sum".

→ It is used to compute response of a discrete-time LTI system with impulse response $h(n)$ for any arbitrary sequence $x(n)$.

→ Any input $x(n)$ can be represented in terms of its unit-impulse sequence $\delta(n)$ which can be represented by

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n-k)$$

The convolution sum is also called superposition sum & it is given as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

Properties of Convolution Integral:

The convolution integral has following properties.

- 1) Commutative: $x(t) * h(t) = h(t) * x(t)$
- 2) Associative: $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$
- 3) Distributive law: $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$

Procedure for computing convolution sum of two discrete time signals:

Step 1: choose initial value of n at evaluation. if $x(n)$ starts at $n=n_1$ & $h(n)$ starts at $n=n_2$ then $n = n_1 + n_2$ is chosen.

Step 2: Express both the sequence in terms of k .

Step 3: Fold $h(k)$ about $k=0$ to obtain $h(-k)$.

Step 4: Shifting of folding sequence. shift $h(-k)$ by n to right if n is positive & left if n is negative to obtain $h(n-k)$

Step 5: Multiply the two sequence $x(k)$ & $h(n-k)$ element by element & sum the product to get $y(n)$.

Step 6: Increment index n , shift the sequence $h(n-k)$ to right by one sample & do step 5.

Step 7: Repeat step 6 until the sum of the products is zero for all remaining values of n .

Unit Step Response of LTI System:

* Unit step response is the output of a LTI system (either continuous time or discrete time) for input is equal to unit-step function or sequence.



Properties of Convolution Integral:

The convolution integral has following properties.

- 1) Commutative : $x(t) * h(t) = h(t) * x(t)$
- 2) Associative : $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$
- 3) Distributive Law : $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$

Procedure for computing convolution sum of two discrete

Time signal:

Step 1: choose initial value of n of evaluation. if $x(n)$ starts at $n=n_1$ & $h(n)$ starts $n=n_2$ then $n = n_1 + n_2$ is good choice.

Step 2: Express both the sequence in terms of k .

Step 3: Fold $h(k)$ about $k=0$ to obtain $h(-k)$.

Step 4: Shifting of folding sequence. Shift $h(-k)$ by n to right if n is positive & left if n is negative to obtain $h(n-k)$.

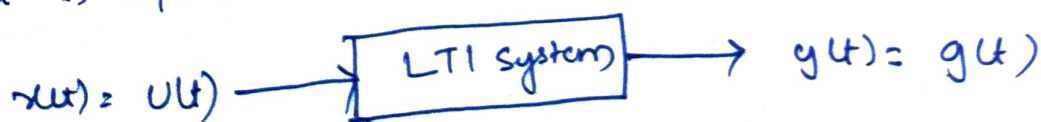
Step 5: Multiply the two sequence $x(k)$ & $h(n-k)$ element by element & sum the product to get $y(n)$.

Step 6: Increment index n , shift the sequence $h(n-k)$ to right by one sample & do step 5.

Step 7: Repeat step 6 until the sum of the products is zero for all remaining values of n .

Unit Step Response of LTI System:

* Unit step response is the output of a LTI system (either continuous time or discrete time) for input is equal to unit-step function or sequence.



Unit step response of continuous time LTI system is determined by convolution integral of $u(t)$ with unit impulse response $h(t)$. is given by

$$g(t) = u(t) * h(t) = h(t) * u(t) \quad \text{by commutative property.}$$

Therefore, unit step response $g(t)$ can be viewed as the response to the input $h(t)$ of a continuous time LTI system with unit impulse response $u(t)$.

- Similarly for discrete time LTI system unit response is given by

$$g(n) = u(n) * h(n) = h(n) * u(n)$$

- So unit step response can also be used to characterize an LTI system.

Problems:

1. Compute the continuous convolution of the following functions:

$$x(t) = e^{-t} u(t)$$

$$h(t) = 2e^{2t} u(t)$$

Ans: We know that $y(t) = x(t) * h(t)$

$$\Rightarrow y(t) = \int x(z) h(t-z) dz$$

for $t \geq 0$

$$\Rightarrow y(t) = \int e^{-z} \cdot 1 \cdot 2 \cdot e^{2(t-z)} \cdot 1 dz$$

$$\Rightarrow y(t) = 2 \int e^{-z} \cdot e^{2(t-z)} dz$$

$$= 2 \int e^{2t-3z} dz$$

$$\Rightarrow y(t) = 2 e^{2t} \int e^{-3\tau} d\tau$$

$$\Rightarrow y(t) = 2 e^{2t} \left(-\frac{1}{3} \right) e^{-3\tau} \Big|_0^t$$

$$\Rightarrow y(t) = -\frac{2}{3} e^{2t} (e^{-3t} - 1)$$

$$\text{for } t < 0, y(t) = 0.$$

② Compute discrete convolution of the following sequences.

$$x(n) = [1, 2, 3]$$

$$h(n) = [2, -1, 3, 0]$$

Ans. $y(n) = \sum x(k) h(n-k)$

Let's calculate step by step.

$$x(k) = [1, 2, 3]$$

$$h(k) = [2, -1, 3, 0]$$

$$h(-k) = [0, 3, -1, 2]$$

for $n=0$

$$y(0) = \sum x(k) \cdot h(-k)$$

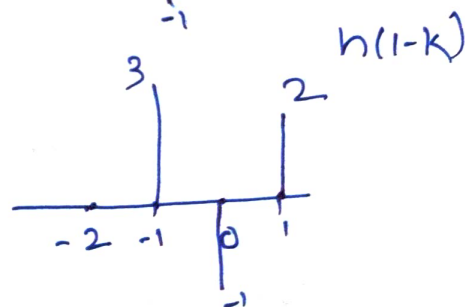
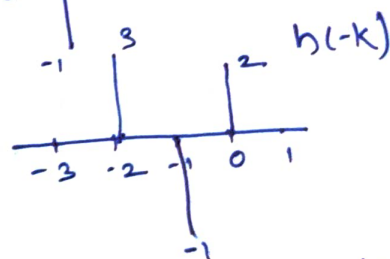
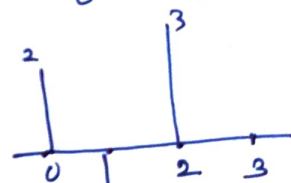
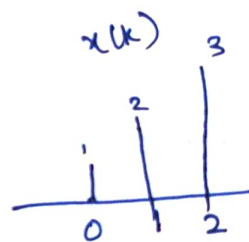
$$= 1 \cdot 2 = 2$$

for $n=1$

$$y(1) = \sum x(k) h(1-k)$$

$$= 1 \cdot (-1) + 2 \cdot 2$$

$$= -1 + 4 = 3$$



for $n=2$

$$y(2) = \sum x(k) h(2-k)$$

$$= 3 \cdot 1 + 2 \cdot (-1) + 3 \cdot (2)$$

$$= 3 - 2 + 6 = 7$$

for $n=3$

$$y(3) = \sum x(k) h(3-k)$$

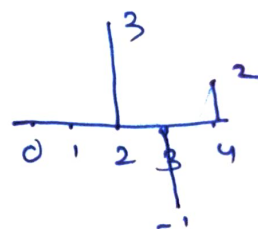
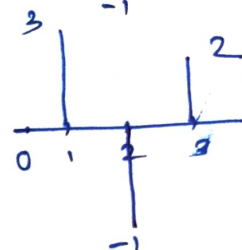
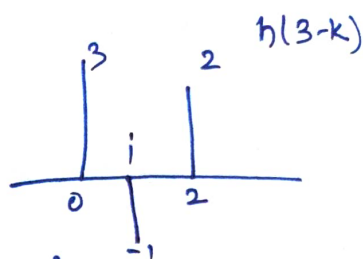
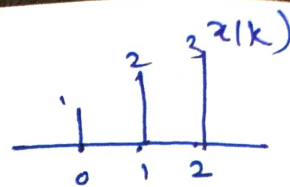
$$= 0 + 2 \cdot 3 + 3 \cdot (-1) + 0$$

$$= 6 - 3 = 3$$

for $n=4$

$$y(4) = \sum x(k) h(4-k)$$

$$= 3 \cdot 3 = 9$$

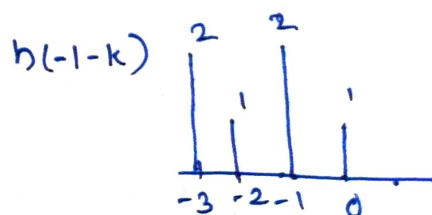
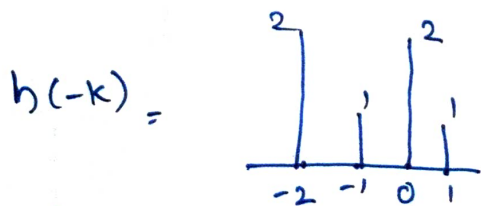
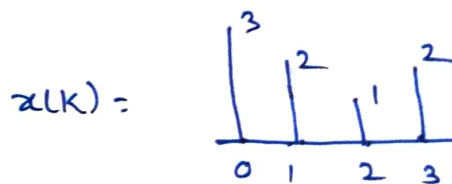
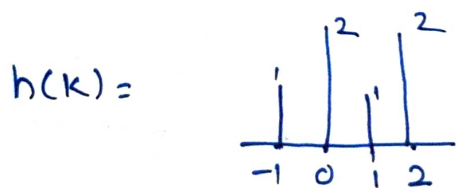


Therefore, the discrete convolution of $x(n)$ and $h(n)$ is

$$y(n) = [2, 3, 7, 3, 9] \quad y(n) = \{2, 3, 7, 3, 9\}$$

③ $x(n) = \{3, 2, 1, 2\}$, $h(n) = \{1, 2, 1, 2\}$ find convolution

Ans: $n_1 = 0$, $n_2 = -1$, $n = 0 + (-1) = -1$



for $n=-1$

$$y(-1) = \sum x(k) h(-1-k)$$

:

Ques

①

		$x(k)$		
		1	2	3
$h(k)$	2	2	4	6
	-1	-1	-2	-3
	3	3	6	9
	0	0	0	0

$$y(n) = \{2, 3, 7, 3, 9, 0\}$$

②

$$x(n) = \{3, 2, 1, 2\}$$

$$h(n) = \{1, 2, 1, 2\}$$

		$x(k)$			
		3	2	1	2
$h(k)$	1	3	2	1	2
	2	6	4	2	4
	1	3	2	1	2
	2	6	4	2	4

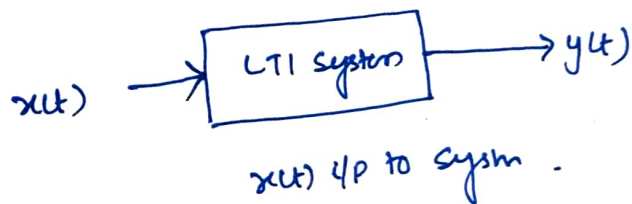
$$y(n) = \{3, 8, 8, 12, 9, 4, 4\}$$

(same as mentioned convolution)

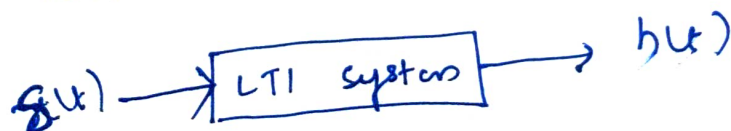
Input Output behaviour with aperiodic convergent inputs.

⇒ Response of a continuous time system:

if an ip $x(t)$ is applied to the output of the system $y(t)$

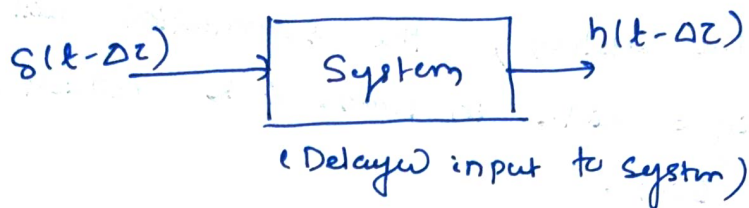


So instead of $x(t)$ we apply a standard elementary signal that is known as impulse signal. If we apply impulse ~~response~~ signal then the system's ~~output~~ response will be impulse response if delta is replaced by Tau delta.



~~if the input is $S(t - \Delta t)$, so the input is delayed~~

if input is $S(t - \Delta t)$, so when input is delayed output will be delayed, this is time invariant system.



$$y(t) = \int_{-\infty}^{\infty} x(z) h(t-z) dz$$

So this equation this convolution in integral equation is very useful for calculating the response of the system.

ii) Response of discrete time system?

if input $x(n)$ is given to discrete time system & this discrete time system is giving output as $y(n)$

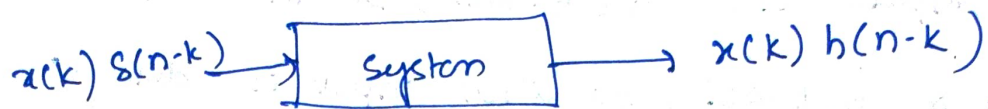


- if this input is an impulse signal $\delta(n)$, then the system will provide output corresponding to impulse signal the output corresponding to impulse signal is called impulse response $h(n)$.



- if the input signal is delayed by k , i.e. $S(n-k)$ & it is applied to the same discrete time system DTS then the output will be delayed by k i.e. $h(n-k)$.

→ if another input $x[k]$ is multiplied with $S(n-k)$ & provided to the system, because it is a linear time invariant system so multiplying is a scaling of input which will also result in scaling of output hence the output will become $x[k] h(n-k)$.



- Dividing summation of the input from $-\infty$ to $+\infty$ & then passing through the system, the summation will also result in the right side in terms of k which is called convolution sum.
- Convolution sum is important for calculating output of the discrete time system.

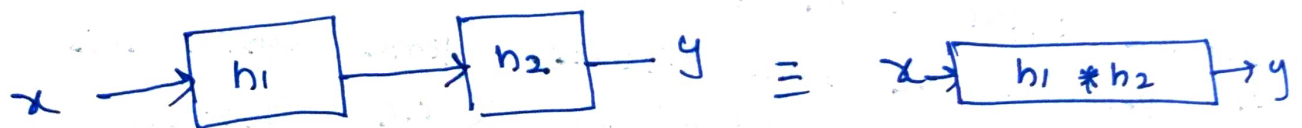
$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$\Rightarrow y(n) = x(n) * h(n)$$

Cascade interconnection:

- Let in a LTI system input is x , output y & impulse response h .
- We know that x & y are related as $y = x * h$
- So here the system is performing convolution operation.
- From the properties of convolution several equivalences involving the impulse response of series & parallel interconnected system.

i) Series Connection:



Let two LTI system responses h_1 & h_2 that are connected in series configuration.

$$y = (x * h_1) * h_2$$

- Due to associative convolution. $y = x * (h_1 * h_2)$

- Thus the series interconnection of two LTI systems behaves as a single LTI system with impulse response $h_1 * h_2$.

on



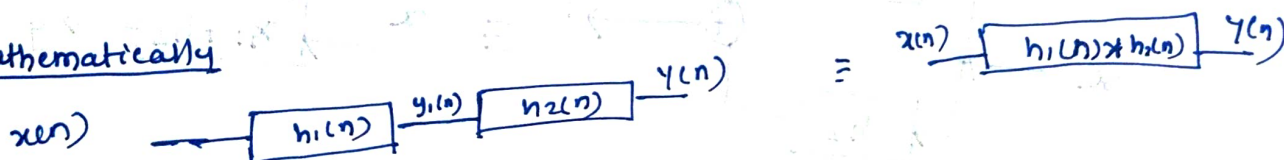
- Here also two LTI systems with impulse response h_1 & h_2 that are connected in series. So $y = (x * h_1) * h_2$

- And Due to associative & commutative convolution law

$$y = x * (h_1 * h_2) = x * (h_2 * h_1) = (x * h_2) * h_1$$

Thus the interchanging the two LTI systems does not change the behavior of the overall system with input x & output y .

Mathematically



Let $y_1(n)$ is the output of 1st system.

$$y_1(k) = x_1(k) * h_1(k) = \sum_{v=-\infty}^{\infty} x(v) h_1(k-v)$$

$$\begin{aligned} \text{The output } y(n) &= y_1(k) * h_2(k) \\ &= \left[\sum_{v=-\infty}^{\infty} x(v) h_1(k-v) \right] * h_2(k) \\ &= \sum_{k=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} x(v) h_1(k-v) \cdot h_2(n-k) \end{aligned}$$

Let $k-v = p$.

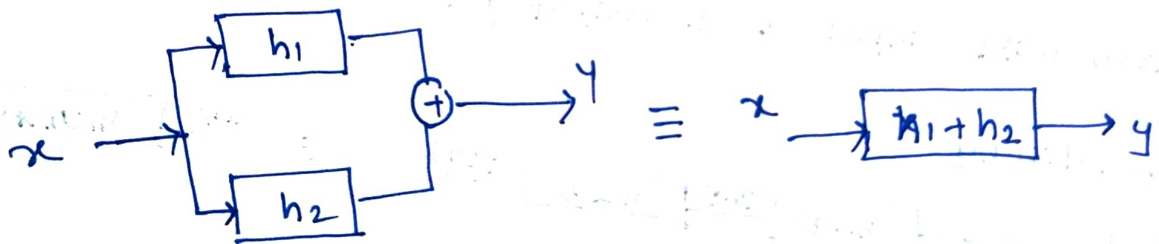
$$\begin{aligned}
 y(n) &= \sum_{v=-\infty}^{\infty} x(v) \sum_{k=-\infty}^{\infty} h_1(k-v) h_2(n-k) \\
 &= \sum_{v=-\infty}^{\infty} x(v) \sum_{p=-\infty}^{\infty} h_1(p) h_2(n-v-p) \\
 &= \sum_{v=-\infty}^{\infty} x(v) h(n-v)
 \end{aligned}$$

$$\Rightarrow \boxed{y(n) = x(n) * h}$$

$$\begin{aligned}
 \text{Where } h(n) &= \sum_{k=-\infty}^{\infty} h_1(k) \cdot h_2(n-k) \\
 &= h_1(n) * h_2(n)
 \end{aligned}$$

11) Parallel Connection:

Consider two LTI system with impulse response h_1 & h_2 are connected parallel configuration as shown below.



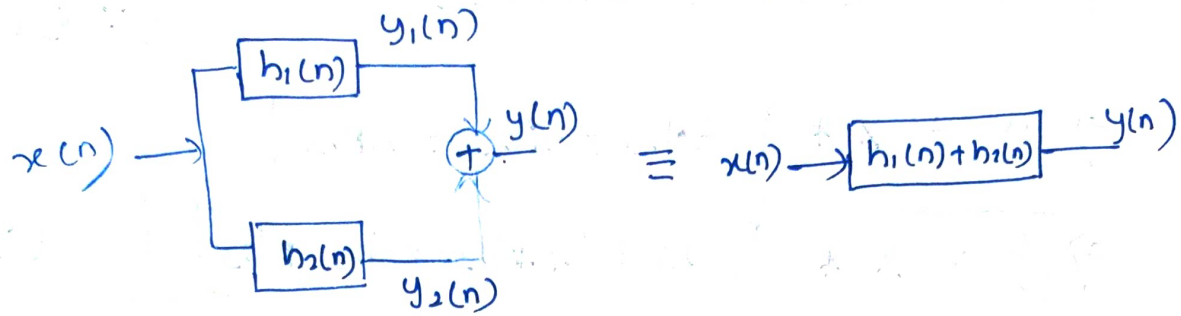
$$\text{Here } \boxed{y = x * h_1 + x * h_2}$$

due to distributive Law of convolution,

$$\boxed{y = x * [h_1 + h_2]}$$

Thus the parallel interconnection of two LTI system behaves as a single LTI system with impulse response $h_1 + h_2$.

Mathematically:-



From this figure

$$y_1(n) = x(n) * h_1(n)$$

$$y_2(n) = x(n) * h_2(n)$$

$$\text{The output } y(n) = y_1(n) + y_2(n)$$

$$= x(n) * h_1(n) + x(n) * h_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k) h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k) h_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(k) [h_1(n-k) + h_2(n-k)]$$

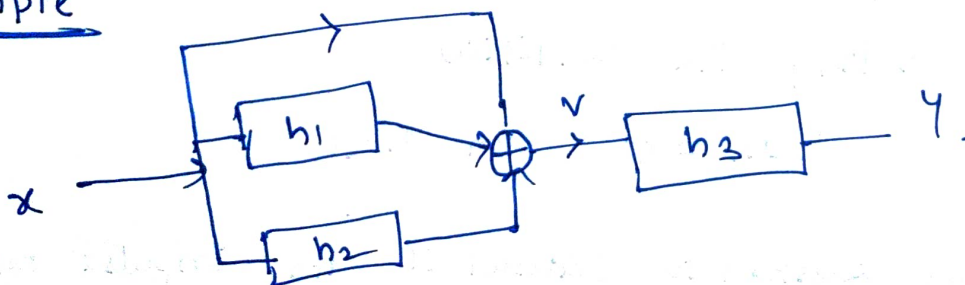
$$= \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$\begin{aligned} & \because h_1(n-k) + h_2(n-k) \\ & = h(n) \\ & h(n) = h_1(n) + h_2(n) \end{aligned}$$

$$\Rightarrow \boxed{y(n) = x(k) * h(n)}$$

Thus if two systems are connected in parallel, then the overall impulse response is equal to sum of two ~~input~~ impulse response.

Example



From this fig $y = v * h_3$

$$\Rightarrow y = (x * [\delta + h_1 + h_2]) * h_3$$

$$(\because v_3 = x * [\delta + h_1 + h_2])$$

$$\Rightarrow y = x * [\delta * h_3 + h_1 * h_3 + h_2 * h_3]$$

$$\Rightarrow \boxed{y = x * [h_3 + h_1 * h_3 + h_2 * h_3]} \quad (\because \delta = 1)$$

Thus the impulse response h of the overall system is

$$\boxed{h(k) = h_3 + h_1 * h_3 + h_2 * h_3}$$

Causality for LTI System:

→ A system is causal if its output depends only on the past and present value of the input signal.

→ For a discrete time LTI system, this requirement in $y(n)$ should not depend on $x(k)$ for $k > n$.

→ Based on the convolution sum equation, all the coefficients $h(n-k)$ that multiplies $x(k)$ for $k > n$ must be zero, which means that the impulse response of a causal discrete-time LTI system should satisfy the condition

$$h(n) = 0 \quad \text{for } n < 0$$

- A causal system is causal if its impulse response is zero for negative time. Thus the system should not have a response before impulse applied.

Mathematically

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$$

$$= \sum_{k=-\infty}^{-1} h(k) x(n-k) + \sum_{k=0}^{\infty} h(k) x(n-k)$$

$$= \underbrace{\dots h(-2)x(n+2) + h(-1)x(n+1)}_{\text{depends on future i/p}} + \underbrace{h(0)x(n)}_{\text{Present i/p}} + \underbrace{h(1)x(n-1) + \dots}_{\text{depends on past i/p}}$$

So

$$y(n) = \sum_{k=0}^{\infty} h(k) x(n-k)$$

Stability

An LTI System is stable if it produces a bounded output sequence for every bounded input sequence.

→ Let $x(n)$ be a bounded input sequence, $h(n)$ be the impulse response of the system & $y(n)$ be the output sequence.

Taking the magnitude of the output

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k) x(n-k) \right|$$

~~Let the bounded value of the i/p is~~

We know that the magnitude of the sum of terms is less than or equal to sum of the magnitudes, hence

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)| |x(n-k)|$$

Let the Bounded value of input is equal to B,

$$|y(n)| \leq B \sum_{k=-\infty}^{\infty} |h(k)|$$

The above condition will be satisfied when

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

So the necessary & sufficient condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Ex Test the stability of system whose impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$

Ans. The necessary & sufficient condition of stability is $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

Given that $h(n) = \left(\frac{1}{2}\right)^n u(n)$

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |h(n)| &= \sum_{n=-\infty}^{\infty} \left| \left(\frac{1}{2}\right)^n u(n) \right| \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2 < \infty\end{aligned}$$

Hence the system is stable.

Periodic & Semiperiodic inputs to an LSI System

Frequency Analysis of discrete time signals:

- Any periodic signal $x(t)$ can be represented as weighted sum of exponentials.
- The Fourier Series of a continuous time periodic signal consists of infinite number of frequency components, since frequency range for continuous time signal extends from $-\infty$ to ∞ .
- The discrete time signal is unique over interval $(-\pi, \pi)$ or $(0, 2\pi)$.
- A discrete time signal of fundamental period N can consist of frequency components separated by $\frac{2\pi}{N}$ radians.
- The Fourier Series representation of discrete time periodic signal will contain at most N frequency components.

Introduction:

- Any periodic non-sinusoidal signal could be represented in terms of linear weighted sum of harmonic sinusoidal signals. This representation is called Fourier Series representation.
- The Fourier representation of signals are used for the frequency domain analysis of signals.
- It enables to extract the amplitude & phase of various frequency component present in signals.
- The Fourier representation of aperiodic or non periodic signal is performed by treating them as periodic signals with an infinite fundamental period. In this case, the non periodic signals are represented as a function of frequency called "Fourier Transform".

Trigonometric Form of Fourier Series:

→ In this form of Fourier representation, any periodic signal $x(t)$ with a fundamental period T is represented in terms of trigonometric functions as,

$$x(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos m\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t \quad - (1)$$

Where $\Omega_0 = \frac{2\pi}{T}$ & a_m and b_n are constants

→ $\cos m\Omega_0 t$ & $\sin n\Omega_0 t$ are orthogonal functions over one period for all integer value of $m \neq n$. This implies that

$$\int_a^{a+T} \cos m\Omega_0 t \cdot \sin n\Omega_0 t \, dt = 0 \quad \text{for any arbitrary value of } a.$$

So eqn (1) can be written as

$$x(t) = \frac{1}{2} a_0 + a_1 \cos \Omega_0 t + a_2 \cos 2\Omega_0 t + \dots + b_1 \sin \Omega_0 t + b_2 \sin 2\Omega_0 t + \dots$$

Where $\Omega_0 = 2\pi f_0 = \frac{2\pi}{T}$ = fundamental frequency in rad/sec

f_0 = Fundamental freq. in Hz

$n\Omega_0$ = Harmonic frequencies

a_0, a_n, b_n = Fourier coefficients of trigonometric form

Here $\frac{a_0}{2}$ is a constant representing the average value or dc component of $x(t)$.

The Fourier coefficient can be expressed as

$$\begin{aligned} a_0 &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) \, dt \\ \text{on } a_0 &= \frac{2}{T} \int_0^T x(t) \, dt \end{aligned} \quad \begin{aligned} a_n &= \frac{2}{T} \int_0^T x(t) \cos n\Omega_0 t \, dt \\ b_n &= \frac{2}{T} \int_0^T x(t) \sin n\Omega_0 t \, dt \end{aligned}$$

The integration is over one period i.e. $-T/2$ to $T/2$ or 0 to T .

Conditions for existence of Fourier Series:

The following conditions called as Dirichlet's conditions must be satisfied by the signals for the existence or convergence of Fourier Series.

- 1) The signal $x(t)$ should be absolutely integrable over one period i.e. $\int_0^T |x(t)| dt < \infty$
- 2) The signal $x(t)$ must have finite number of maxima & minima in one period.
- 3) The signal $x(t)$ must have finite number of discontinuities in one period.

If any signal $x(t)$ satisfies the above Dirichlet's conditions, the Fourier series represented by eqn (1) converges i.e. the sum becomes equal to $x(t)$ except at the point of discontinuity.

Derivation equation for a_0 :

$$a_0 = \frac{2}{T} \int_0^T x(t) dt$$

Proof: The Fourier series of $x(t)$ is given by

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

Integrate both side for 0 to T

$$\Rightarrow \int_0^T x(t) dt = \int_0^T \frac{a_0}{2} dt + \int_0^T \sum_{n=1}^{\infty} a_n \cos n\omega t dt + \int_0^T \sum_{n=1}^{\infty} b_n \sin n\omega t dt$$

$$\Rightarrow \int_0^T x(t) dt = \frac{a_0}{2} \int_0^T dt + \sum_{n=1}^{\infty} a_n$$

Derivation equation for a_0 :

$$a_0 = \frac{2}{T} \int_0^T x(t) dt$$

Proof: The Fourier series of $x(t)$ is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Integrate both side ~~from 0 to T~~ for 0 to T

$$\int_0^T x(t) dt = \int_0^T \frac{a_0}{2} dt + \int_0^T \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t dt + \int_0^T \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t dt$$

$$\Rightarrow \int_0^T x(t) dt = \frac{a_0}{2} \int_0^T dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos n\Omega_0 t dt + \sum_{n=1}^{\infty} b_n \int_0^T \sin n\Omega_0 t dt$$

$$\begin{aligned} \Rightarrow \int_0^T x(t) dt &= \frac{a_0}{2} [t]_0^T + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n\Omega_0 t}{n\Omega_0} \right]_0^T + \sum_{n=1}^{\infty} b_n \left[-\frac{\cos n\Omega_0 t}{n\Omega_0} \right]_0^T \\ &= \frac{a_0}{2} [T-0] + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n\Omega_0 T}{n\Omega_0} - \frac{\sin n\Omega_0 0}{n\Omega_0} \right] + \sum_{n=1}^{\infty} b_n \left[-\frac{\cos n\Omega_0 T}{n\Omega_0} + \frac{\cos n\Omega_0 0}{n\Omega_0} \right] \\ &= \frac{T}{2} a_0 + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n \cdot \frac{2\pi}{T} \cdot T}{n \frac{2\pi}{T}} \right] + \sum_{n=1}^{\infty} b_n \left[\frac{-\cos n \cdot \frac{2\pi}{T} \cdot T}{n \frac{2\pi}{T}} + \frac{1}{n \frac{2\pi}{T}} \right] \end{aligned}$$

$$\text{As } \Omega_0 = \frac{2\pi}{T}, \cos 0 = 1, \sin 0 = 0$$

$$\Rightarrow \int_0^T x(t) dt = \frac{T}{2} a_0 + \sum_{n=1}^{\infty} a_n T \left[\frac{\sin n 2\pi}{n 2\pi} \right] + \sum_{n=1}^{\infty} b_n T \left[-\frac{\cos n 2\pi}{n 2\pi} + \frac{1}{n 2\pi} \right]$$

$$\Rightarrow \int_0^T x(t) dt = \frac{T}{2} a_0 + \sum_{n=1}^{\infty} a_n \cdot T \cdot 0 + \sum_{n=1}^{\infty} b_n T \left[-\frac{1}{n 2\pi} + \frac{1}{n 2\pi} \right]$$

$$\Rightarrow \int_0^T x(t) dt = \frac{T}{2} a_0 + 0 + 0 \quad \left(\because \sin n 2\pi = 0 \right. \\ \left. \cos n 2\pi = 1 \right)$$

$$\Rightarrow \int_0^T x(t) dt = \frac{T}{2} a_0$$

$$\Rightarrow \boxed{a_0 = \frac{2}{T} \int_0^T x(t) dt}$$

Proof .

Derivation equation for a_n :

$$\boxed{a_n = \frac{2}{T} \int_0^T x(t) \cos n \omega_0 t dt}$$

Proof : Fourier series of $x(t)$ is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n \omega_0 t + \sum_{n=1}^{\infty} b_n \sin n \omega_0 t$$

Let us multiply above eqn by $\cos m \omega_0 t$ & integrate over 0 to T

$$\Rightarrow \int_0^T x(t) \cos m \omega_0 t dt = \int_0^T \frac{a_0}{2} \cos m \omega_0 t dt + \int_0^T \sum_{n=1}^{\infty} a_n \cos n \omega_0 t \cos m \omega_0 t dt \\ + \int_0^T \sum_{n=1}^{\infty} b_n \sin n \omega_0 t \cdot \cos m \omega_0 t dt .$$

$$\Rightarrow \int_0^T x(t) \cos m \omega_0 t dt = \int_0^T \frac{a_0}{2} \cos m \omega_0 t dt + \int_0^T a_1 \cos \omega_0 t \cdot \cos m \omega_0 t dt \\ + \dots + \int_0^T a_m \cos^2 m \omega_0 t dt + \dots + \int_0^T b_1 \sin \omega_0 t \cdot \cos m \omega_0 t dt + \\ \dots + \int_0^T b_m \sin m \omega_0 t \cdot \cos m \omega_0 t dt + \dots$$

In above eqn all definite integration becomes zero except

$$\int_0^T a_m \cos^2 m \Omega_0 t \, dt$$

$$\Rightarrow \int_0^T x(t) \cos m \Omega_0 t \, dt = \int_0^T a_m \cos^2 m \Omega_0 t \, dt$$

$$\Rightarrow \int_0^T x(t) \cos m \Omega_0 t \, dt = a_m \int_0^T \cos^2 m \Omega_0 t \, dt$$

$$= a_m \int_0^T \frac{(1 + \cos 2m \Omega_0 t)}{2} dt$$

$$= \frac{a_m}{2} \int_0^T (1 + \cos 2m \Omega_0 t) dt$$

$$= \frac{a_m}{2} \left[\int_0^T dt + \int_0^T \cos 2m \Omega_0 t \, dt \right]$$

$$= \frac{a_m}{2} \left[t + \frac{\sin 2m \Omega_0 t}{2m \Omega_0} \right]_0^T$$

$$= \frac{a_m}{2} \left[(T-0) + \frac{\sin 2m \Omega_0 T}{2m \Omega_0} - \frac{\sin 2m \Omega_0 0}{2m \Omega_0} \right]$$

$$= \frac{a_m}{2} \left[T + \frac{\sin 2m \Omega_0 T}{2m \Omega_0} - \frac{\sin 0}{2m \Omega_0} \right]$$

$$= \frac{a_m}{2} \left[T + \frac{\sin 2m \frac{2\pi}{T} \cdot T}{2m \frac{2\pi}{T}} \right]$$

$$= \frac{a_m}{2} \left[T + \frac{\sin 2m 2\pi}{2m \frac{2\pi}{T}} \right]$$

$$= a_m \frac{T}{2}$$

$$(\because \sin 2m\pi = 0)$$

$$\Rightarrow \int_0^T x(t) \cos m \Omega_0 t \, dt = a_m \frac{T}{2}$$

$$\Rightarrow a_m = \frac{2}{T} \int_0^T x(t) \cos m \Omega_0 t \, dt$$

for n th coefficient
 $n = m$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos n \Omega_0 t \, dt$$

Derivation equation for b_n :

$$b_n = \frac{2}{T} \int_0^T x(t) \sin n\omega t \, dt$$

Proof: The trigonometric Fourier Series eqn is given by

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

Let us multiply both side with $\sin m\omega t$ & taking integration over 0 to T.

$$\begin{aligned} \int_0^T x(t) \sin m\omega t \, dt &= \int_0^T \frac{a_0}{2} \sin m\omega t \, dt + \int_0^T \sum_{n=1}^{\infty} a_n \cos n\omega t \cdot \sin m\omega t \, dt \\ &\quad + \int_0^T \sum_{n=1}^{\infty} b_n \sin n\omega t \cdot \sin m\omega t \, dt \\ &= \frac{a_0}{2} \int_0^T \sin m\omega t \, dt + \int_0^T a_1 \cos \omega t \cdot \sin m\omega t \, dt + \dots + \int_0^T a_m \cos m\omega t \cdot \sin m\omega t \, dt \\ &\quad + \dots + \int_0^T b_1 \sin \omega t \sin m\omega t \, dt + \dots + \int_0^T b_m \sin m\omega t \cdot \sin m\omega t \, dt + \dots \end{aligned}$$

In above eqn all the definite integration becomes zero except $\int_0^T b_m \sin^2 m\omega t \, dt$

$$\text{Hence } \int_0^T x(t) \sin m\omega t \, dt = \int_0^T b_m \sin^2 m\omega t \, dt$$

$$\Rightarrow \int_0^T x(t) \sin m\omega t \, dt = b_m \int_0^T \left(\frac{1 - \cos 2m\omega t}{2} \right) dt$$

$$\Rightarrow \int_0^T x(t) \sin m\omega t \, dt = \frac{b_m}{2} \int_0^T (1 - \cos 2m\omega t) dt$$

$$= \frac{b_m}{2} \left[\int_0^T dt - \int_0^T \cos 2m\omega t \, dt \right]$$

$$= \frac{b_m}{2} \left[(t - \cancel{\cos 2m\omega t}) - \frac{\sin 2m\omega t}{2m\omega} \right]_0^T$$

$$\Rightarrow \int_0^T x(t) \sin m \omega_0 t \, dt = \frac{b_m}{2} \left[(T-0) - \left(\frac{\sin m \omega_0 T}{2m \omega_0} - \frac{\sin 2m \omega_0}{2m \omega_0} \right) \right]$$

$$\Rightarrow \int_0^T x(t) \sin m \omega_0 t \, dt = \frac{b_m}{2} \left[T - \frac{\sin 2m \cdot \frac{2\pi}{T} \cdot T}{2m \frac{2\pi}{T}} \right]$$

$$\Rightarrow \int_0^T x(t) \sin m \omega_0 t \, dt = b_m \cdot \frac{T}{2} \quad (\because \sin 2m 2\pi = 0)$$

$$\Rightarrow b_m = \frac{2}{T} \int_0^T x(t) \sin m \omega_0 t \, dt$$

for $m=n$,

$$\boxed{b_n = \frac{2}{T} \int_0^T x(t) \sin n \omega_0 t \, dt}$$

Prove .

Trigonometric Fourier Series:

Any periodic signal $x(t)$ with fundamental period T is represented in terms of trigonometric Fourier Series as

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

The Fourier coefficient can be expressed as

$$\begin{aligned} a_0 &= \frac{2}{T} \int_0^T x(t) dt \\ a_n &= \frac{2}{T} \int_0^T x(t) \cos n\omega t dt \\ b_n &= \frac{2}{T} \int_0^T x(t) \sin n\omega t dt \end{aligned}$$

Derivation for a_0 :

$$a_0 = \frac{2}{T} \int_0^T x(t) dt$$

Proof: The Fourier Series of $x(t)$ is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

Integrate both side over 0 to T :

$$\int_0^T x(t) dt = \int_0^T \frac{a_0}{2} dt + \int_0^T \sum_{n=1}^{\infty} a_n \cos n\omega t dt + \int_0^T \sum_{n=1}^{\infty} b_n \sin n\omega t dt$$

$$\begin{aligned} \Rightarrow \int_0^T x(t) dt &= \frac{a_0}{2} \int_0^T dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos n\omega t dt + \sum_{n=1}^{\infty} b_n \int_0^T \sin n\omega t dt \\ &= \frac{a_0}{2} t \Big|_0^T + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n\omega t}{n\omega} \right]_0^T + \sum_{n=1}^{\infty} b_n \left[-\frac{\cos n\omega t}{n\omega} \right]_0^T \end{aligned}$$

Derive

$$\Rightarrow \int_0^T x(t) dt = \frac{a_0}{2} [T-0] + \sum_{n=1}^{\infty} a_n \left(\frac{\sin n\omega T}{n\omega} - \frac{\sin n\omega 0}{n\omega} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{-\cos n\omega T}{n\omega} + \frac{\cos n\omega 0}{n\omega} \right)$$

As $\omega = \frac{2\pi}{T}$, $\sin 0 = 0$ & $\cos 0 = 1$, above eqn becomes

$$\Rightarrow \int_0^T x(t) dt = a_0 \frac{T}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{\sin n \frac{2\pi}{T} T}{n \frac{2\pi}{T}} - 0 \right) + \sum_{n=1}^{\infty} b_n \left[\frac{-\cos n \frac{2\pi}{T} T}{n \frac{2\pi}{T}} + \frac{1}{n \frac{2\pi}{T}} \right]$$

$$\Rightarrow \int_0^T x(t) dt = a_0 \frac{T}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{\sin n 2\pi}{n 2\pi} \right) + \sum_{n=1}^{\infty} b_n \left[\frac{-\cos n 2\pi}{n 2\pi} + \frac{1}{n 2\pi} \right]$$

$$\Rightarrow \int_0^T x(t) dt = a_0 \frac{T}{2} + \sum_{n=1}^{\infty} a_n T \left(\frac{\sin n 2\pi}{n 2\pi} \right) + \sum_{n=1}^{\infty} b_n T \left[\frac{-\cos n 2\pi}{n 2\pi} + \frac{1}{n 2\pi} \right]$$

$$\Rightarrow \int_0^T x(t) dt = a_0 \frac{T}{2} + \sum_{n=1}^{\infty} a_n T \cdot 0 + \sum_{n=1}^{\infty} b_n T \left[-\frac{1}{n 2\pi} + \frac{1}{n 2\pi} \right]$$

$$\Rightarrow \int_0^T x(t) dt = a_0 \frac{T}{2} + \sum_{n=1}^{\infty} 0 + \sum_{n=1}^{\infty} b_n T \cdot 0$$

$$\Rightarrow \int_0^T x(t) dt = a_0 \frac{T}{2}$$

$$\Rightarrow \boxed{a_0 = \frac{2}{T} \int_0^T x(t) dt}$$

Prove.

Derivation for a_n :

$$a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega t dt$$

Proof: Fourier Series of $x(t)$ is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

Multiply both side by $\cos m\omega t$ & integrate over 0 to T.

$$\int_0^T x(t) \cos m\omega t dt = \int_0^T \frac{a_0}{2} \cos m\omega t dt + \int_0^T \sum_{n=1}^{\infty} a_n \cos n\omega t \cdot \cos m\omega t dt + \int_0^T \sum_{n=1}^{\infty} b_n \sin n\omega t \cdot \cos m\omega t dt$$

$$\Rightarrow \int_0^T x(t) \cos m\omega t dt = \frac{a_0}{2} \int_0^T \cos m\omega t dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos n\omega t \cdot \cos m\omega t dt + \sum_{n=1}^{\infty} b_n \int_0^T \sin n\omega t \cdot \cos m\omega t dt$$

$$= \frac{a_0}{2} \int_0^T \cos m\omega t dt + a_1 \int_0^T \cos \omega t \cdot \cos m\omega t dt + \dots + a_m \int_0^T \cos m\omega t \cdot \cos m\omega t dt + \dots + b_1 \int_0^T \sin \omega t \cdot \cos m\omega t dt + \dots + b_m \int_0^T \sin m\omega t \cdot \cos m\omega t dt + \dots$$

In above eqn all definite integration becomes zero except

$$\int_0^T \cos^2 m\omega t dt$$

$$\Rightarrow \int_0^T x(t) \cos m\omega t dt = a_m \int_0^T \cos^2 m\omega t dt$$

$$\begin{aligned}
\Rightarrow \int_0^T x(t) \cos m\omega t \, dt &= am \int_0^T \frac{(1 + \cos 2m\omega t)}{2} \, dt \\
&= \frac{am}{2} \left[\int_0^T dt + \int_0^T \cos 2m\omega t \, dt \right] \\
&= \frac{am}{2} \left[\left[t \right]_0^T + \left[\frac{\sin 2m\omega t}{2m\omega} \right]_0^T \right] \\
&= \frac{am}{2} \left[(T-0) + \frac{\sin 2m\omega T}{2m\omega} - \frac{\sin 2m\omega \cdot 0}{2m\omega} \right] \\
&= \frac{am}{2} \left[T + \frac{\sin 2m \frac{2\pi}{T} T}{2m \frac{2\pi}{T}} \right] \\
&= \frac{am}{2} T \quad (\because \sin 2m 2\pi = 0)
\end{aligned}$$

$$\Rightarrow \int_0^T x(t) \cos m\omega t \, dt = am \frac{T}{2}$$

$$\Rightarrow am = \frac{2}{T} \int_0^T x(t) \cos m\omega t \, dt$$

for n th coefficient on when $m=n$,

$$a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega t \, dt$$

Derivation for b_n :

$$b_n = \frac{2}{T} \int_0^T x(t) \sin n\omega t dt$$

Proof: The trigonometric Fourier series eqn is given by

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

Multiply both side by $\sin m\omega t$ & take integration over 0 to T.

$$\int_0^T x(t) dt = \int_0^T \frac{a_0}{2} dt + \int_0^T \sum$$

$$\int_0^T x(t) \cdot \sin m\omega t dt = \int_0^T \frac{a_0}{2} \sin m\omega t dt + \int_0^T \sum_{n=1}^{\infty} a_n \cos n\omega t \cdot \sin m\omega t dt + \int_0^T b_n \sin n\omega t \cdot \sin m\omega t dt$$

$$\Rightarrow \int_0^T x(t) \sin m\omega t dt = \frac{a_0}{2} \int_0^T \sin m\omega t dt + \int_0^T a_1 \cos \omega t \cdot \sin m\omega t dt + \dots + \int_0^T a_m \cos m\omega t \cdot \sin m\omega t dt + \dots + \int_0^T b_1 \sin \omega t \cdot \sin m\omega t dt + \dots + \int_0^T b_m \sin m\omega t \cdot \sin m\omega t dt + \dots$$

In above equation all the definite integration becomes zero except $\int_0^T b_m \sin^2 m\omega t dt$

Hence above equation becomes

$$\Rightarrow \int_0^T x(t) \sin m\omega t dt = \int_0^T b_m \sin^2 m\omega t dt$$

$$= b_m \int_0^T \left(\frac{1 - \cos 2m\omega t}{2} \right) dt$$

$$= \frac{b_m}{2} \left[\int_0^T dt - \int_0^T \cos 2m\omega t dt \right]$$

$$\begin{aligned}
 \Rightarrow \int_0^T x(t) \sin m\omega t \, dt &= \frac{bm}{2} \left[\left[t \right]_0^T - \frac{\sin 2m\omega t}{2m\omega} \right]_0^T \\
 &= \frac{bm}{2} \left[(T-0) - \left(\frac{\sin 2m\omega T}{2m\omega} - \frac{\sin 2m\omega \cdot 0}{2m\omega} \right) \right] \\
 &= \frac{bm}{2} \left[T - \left(\frac{\sin 2m \frac{2\pi}{T} T}{2 \frac{2\pi}{T} m} - 0 \right) \right] \\
 &= \frac{bm}{2} [T - 0] \quad (\because \sin 2m2\pi = 0)
 \end{aligned}$$

$$\Rightarrow \int_0^T x(t) \sin m\omega t \, dt = bm \frac{T}{2}$$

$$\Rightarrow bm = \frac{2}{T} \int_0^T x(t) \sin m\omega t \, dt$$

for $m=n$,

$$\Rightarrow \boxed{b_n = \frac{2}{T} \int_0^T x(t) \sin m\omega t \, dt}$$

Ans.

Even Function or signal:

- A signal or function $x(t)$ is said to be even if $x(t) = x(-t)$.

- In other words if one signal & its mirror image are equal then it is called an even signal.

Ex: cosine signal.

- In even functions, the Fourier Series coefficients

b_n are absent.

• ~~Determine the trigon~~

The Fourier coefficients are given by

$$a_0 = \frac{4}{T} \int_0^{T/2} x(t) dt$$

$$a_n = \frac{4}{T} \int_0^{T/2} x(t) \cos n\omega t dt$$

$$b_n = 0$$

Odd function Signal:

$x(t)$ is an odd signal if it satisfies condition $x(-t) = -x(t)$

The Fourier coefficients are given by

$$a_0 = 0$$

$$a_n = 0$$

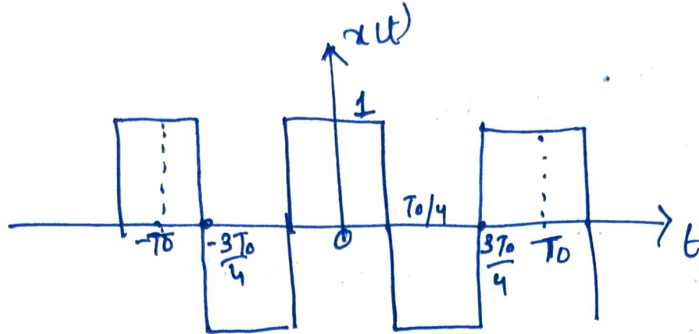
$$b_n = \frac{4}{T} \int_0^{T/2} x(t) \sin n\omega t dt$$

Halfwave Symmetry:

A periodic signal or function is said to be half wave symmetric if

$$x(t) = -x(t \pm T/2)$$

① Find the trigonometric Fourier Series representation of the continuous time-wave voltage signal.



Ans The analytical form of the signal $x(t)$ is given by

$$\begin{aligned} x(t) &= 1 \quad \text{for } 0 < t < T_0/4 \\ &= -1 \quad \text{for } T_0/4 < t < 3T_0/4 \\ &= 1 \quad \text{for } 3T_0/4 < t < T_0 \end{aligned}$$

In this square wave, the average value over one period is zero i.e. $A_0 = 0$

$$A_n = \frac{2}{T} \int_0^{T_0} x(t) \cos n\omega t \, dt$$

$$= \frac{2}{T_0} \left[\int_0^{T_0/4} 1 \cdot \cos n\omega t \, dt + \int_{T_0/4}^{3T_0/4} (-1) \cos n\omega t \, dt + \int_{3T_0/4}^{T_0} 1 \cdot \cos n\omega t \, dt \right]$$

$$= \frac{2}{T_0} \left[\left[\frac{\sin n\omega t}{n\omega} \right]_0^{T_0/4} - \left[\frac{\sin n\omega t}{n\omega} \right]_{T_0/4}^{3T_0/4} + \left[\frac{\sin n\omega t}{n\omega} \right]_{3T_0/4}^{T_0} \right]$$

$$+ \left[\frac{\sin n \frac{2\pi}{T_0} \cdot T_0}{n \frac{2\pi}{T_0}} - \frac{\sin n \frac{2\pi}{T_0} \frac{3T_0}{4}}{n \frac{2\pi}{T_0}} \right] \frac{\sin n \frac{2\pi}{T_0} \cdot T_0}{n \frac{2\pi}{T_0}}$$

$$= \frac{2}{T_0} \left[\frac{\sin n \omega \cdot \frac{T_0}{4}}{n \cdot \frac{2\pi}{T_0}} - \frac{\sin n \omega 0}{\frac{2\pi}{T_0}} \right] - \left[\frac{\sin n \omega \frac{3T_0}{4}}{n \frac{2\pi}{T_0}} \right]$$

$$= \frac{2}{T_0} \left[\frac{1}{n\omega} \left[\sin n \omega \frac{T_0}{4} - \sin n \omega 0 \right] - \frac{1}{n\omega} \left[\sin n \omega \frac{3T_0}{4} - \sin n \omega \frac{T_0}{4} \right] \right. \\ \left. + \frac{1}{n\omega} \left[\sin n \omega T_0 - \sin n \omega \frac{3T_0}{4} \right] \right]$$

$$= \frac{2}{T_0} \times \frac{1}{n\omega} \left[\sin n \omega \frac{T_0}{4} - \sin n \omega \frac{3T_0}{4} + \sin n \omega \frac{T_0}{4} \right. \\ \left. + \sin n \omega T_0 - \sin n \omega \frac{3T_0}{4} \right]$$

$$= \frac{2}{T_0} \times \frac{1}{\frac{2\pi n}{T_0}} \left[2 \sin \left(n \omega \frac{T_0}{4} \right) - 2 \sin \left(n \omega \frac{3T_0}{4} \right) + \sin n \omega T_0 \right]$$

$$= \frac{2}{T_0} \times \frac{T_0}{2\pi n} \left[2 \sin \left(n \omega \frac{T_0}{4} \right) - 2 \sin \left(n \omega \frac{3T_0}{4} \right) + 0 \right]$$

$$\because \sin(n\omega T_0) = 0 \\ \text{for } n\omega T_0 = 2\pi$$

$$= \frac{1}{\pi n} \left[2 \sin \left(n \omega \frac{T_0}{4} \right) - 2 \sin \left(n \omega \frac{3T_0}{4} \right) \right]$$

$$= \frac{2}{\pi n} \left[\sin \left(n \omega \frac{T_0}{4} \right) - \sin \left(n \omega \frac{3T_0}{4} \right) \right]$$

The Fourier Transform:

- The Fourier Transform is a mathematical tool to represent a time domain signal into frequency domain signal.

→ The Fourier transform of signal $x(t)$ is defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\Rightarrow X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt = F[x(t)]$$

where f = cyclic frequency.

Condition for Fourier Transform Existence:

The Fourier transform exist for the signal $x(t)$ if it follows Dirichlet condition.

1. The signal $x(t)$ is absolute integrable.

$$\int_{-\infty}^{\infty} x(t) dt < \infty$$

2. The signal $x(t)$ should have a finite number of maxima & minima within a finite duration interval.

3. The signal $x(t)$ can have a finite number of discontinuities within an interval.

Inverse Fourier Transform:

The inverse Fourier transform of $X(\omega)$ is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j2\pi f t} d\omega = F^{-1}[X(\omega)]$$

X Magnitude & Phase Spectrum Using Fourier Transform:

Let the Fourier transform $X(j\omega)$ is a complex function of frequency ω & it can be defined in terms of real & imaginary part.

$$X(j\omega) = X_{\text{real}}(j\omega) + jX_{\text{imag}}(j\omega)$$

The magnitude spectrum of $x(j\omega)$ can be written as

$$|X(j\omega)| = \sqrt{X_{\text{real}}^2(j\omega) + X_{\text{imag}}^2(j\omega)}$$

The phase spectrum of $x(j\omega)$ can be written as

$$\angle X(j\omega) = \tan^{-1} \frac{X_{\text{imag}}(j\omega)}{X_{\text{real}}(j\omega)}$$

→ Magnitude spectrum has always even symmetry whereas phase spectrum has always odd symmetry.

→ Both magnitude spectrum & phase spectrum are called frequency spectrum.

Convolution / Multiplication & their effect in frequency domain, Magnitude & Phase Response, Fourier domain duality.

Convolution Property:

→ The Fourier transform of $x_1(t)$ is $X_1(j\omega)$

The Fourier transform of $x_2(t)$ is $X_2(j\omega)$ Then

the property of convolution of CTFT

$$x_1(t) * x_2(t) \xrightarrow{\text{CTFT}} X_1(j\omega) \cdot X_2(j\omega)$$

Multiplication Property of CTFT:

→ Due to duality property betⁿ Time & frequency domains, the multiplication in time domain corresponds to convolution in the frequency domain.

$$r(t) = p(t) \cdot q(t) \xrightarrow{\text{CTFT}} R(j\omega) = \frac{1}{2\pi} [P(j\omega) * Q(j\omega)]$$

Duality

By comparing Fourier Transform & inverse Fourier transform relations given as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{--- (1)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{--- (2)}$$

Above two eqⁿ are similar but not identical in forms. This Symmetric property of FT is called duality.

Magnitude & Phase Response of CT LTI System:

The output of LTI system in terms of convolution equation

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(z) x(t-z) dz$$

$$\text{Let } x(t) = A e^{j\omega t}$$

$$\Rightarrow x(t-z) = A e^{j\omega(t-z)}$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} h(z) A e^{j\omega(t-z)} dz$$

$$= \int_{-\infty}^{\infty} h(z) A e^{j\omega t} \cdot e^{-j\omega z} dz$$

$$= A e^{j\omega t} \int_{-\infty}^{\infty} h(z) e^{-j\omega z} dz$$

From FT,

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} h(z) e^{-j\omega z} dz$$

So $y(t)$ can be written as

$$y(t) = x(t) H(j\omega)$$

From above eqⁿ, if a complex signal is given as input signal to CT LTI system, then the output has the same frequency as that of input signal multiply by $H(j\omega)$. Hence $H(j\omega)$ is called frequency response of CT LTI system.

The $H(j\omega)$ can be represented in terms of magnitude $\times$ phase as

$$H(j\omega) = |H(j\omega)| \angle H(j\omega)$$

↙
Magnitude Functn.

↘
Phase Functn

$$H(j\omega) = H_{\text{real}}(j\omega) + j H_{\text{imaginary}}(j\omega)$$

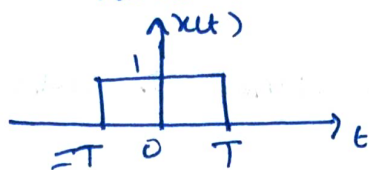
The magnitude function is written as

$$|H(j\omega)| = \sqrt{H_{\text{real}}^2(j\omega) + H_{\text{imaginary}}^2(j\omega)}$$

The phase function is written as

$$\angle H(j\omega) = \tan^{-1} \frac{H_{\text{imaginary}}(j\omega)}{H_{\text{real}}(j\omega)}$$

2. Determine the Fourier Transform of the rectangular pulse



Ans. From the fig. $x(t) = 1$ for $-T < t < T$

Using FT

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-T}^T 1 \cdot e^{-j\omega t} dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-T}^T$$

$$= \frac{e^{-j\omega T}}{-j\omega} - \frac{e^{-j\omega(-T)}}{-j\omega}$$

$$= \frac{e^{-j\omega T}}{-j\omega} + \frac{e^{j\omega T}}{j\omega}$$

$$= \frac{1}{j\omega} [e^{j\omega T} - e^{-j\omega T}]$$

$$= \frac{1}{j\omega} 2j \sin \omega T$$

$$= 2 \frac{\sin \omega T}{\omega}$$

$$\Rightarrow X(\omega) = 2T \frac{\sin \omega T}{\omega T} \quad (\because \text{multiply } T \text{ bot Numerator \& Denom})$$

$$\Rightarrow X(\omega) = 2T \text{sinc } \omega T \quad (\because \frac{\sin \theta}{\theta} = \text{sinc } \theta)$$

Ans.

The Discrete Time Fourier Transform (DTFT) Representation

→ The FT of discrete time signal is called Discrete time Fourier Transform (DTFT).

Let $x(n)$ = finite discrete time signal.

$X(e^{j\omega})$ = DTFT of signal $x(n)$

$$X(e^{j\omega}) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

~~givent DTFT:~~

~~$$x(n) = F^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$~~

~~The Discrete~~

Ex: Determine DTFT of signal $x(n) = A^n u(n)$.

Ans $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$$= \sum_{n=0}^{\infty} A^n u(n) e^{-j\omega n} = \sum_{n=0}^{\infty} A^n \cdot 1 \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} (A e^{-j\omega})^n$$

$$= \frac{1}{1 - A e^{-j\omega}}$$

→ $X(e^{j\omega}) = \frac{1}{1 - A e^{-j\omega}}$

summary

The Discrete Fourier Transform (DFT):

Let $X[e^{j\omega}]$ be DTFT of the discrete time signal $x(n)$.

The discrete Fourier Transform (DFT) ~~$x(n)$~~ $x(n)$ is calculated by sampling one period of the DTFT $X[e^{j\omega}]$ at a finite number of frequency points.

Let one period consists of N equally spaced points, $0 \leq \omega \leq 2\pi$.

Each frequency point is represented by ratio,

$$\omega_k = \frac{2\pi k}{N} \quad \text{for } k = 0, 1, 2, \dots, N-1$$

Hence the sampling of DTFT at frequency points is written as

$$X[k] = X[e^{j\omega}]_{\omega_k = \frac{2\pi k}{N}} \quad \text{for } k = 0, 1, 2, \dots, N-1$$

DFT is also called as N point DFT, where the number of samples N for a finite duration sequence $x(n)$ of length L should be such that $N \geq L$, in order to avoid aliasing.

Let $x(n)$ = Discrete time signal having length as L

$$X(k) = \text{DFT}[x(n)]$$

Hence a N point DFT of $x(n)$ is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi n k}{N}}$$

$$\text{for } k = 0, 1, 2, \dots, N-1$$

Inverse DFT:

~~X(k)~~ The inverse DFT of $x(k)$ is defined as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j \frac{2\pi nk}{N}}$$

Parseval's Theorem:

$$\text{If } \text{DFT}[x_1(n)] = X_1(k)$$

$$\text{DFT}[x_2(n)] = X_2(k)$$

Then

$$\sum_{n=0}^{N-1} x_1(n) \cdot x_2^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2^*(k)$$

If $x_1(n) = x_2(n)$ then $X_1(k) = X_2(k) = X(k)$.

Then above theorem will be

$$\sum_{n=0}^{N-1} x(n) \cdot x^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot X^*(k)$$

But we know that $x(n) \cdot x^*(n) = |x(n)|^2$
& $X(k) \cdot X^*(k) = |X(k)|^2$

Putting these values in above expression, we get

$$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |X(k)|^2$$

When $x_1(n) = x_2(n)$ & $X_1(k) = X_2(k)$, it is called
Special form of Parseval's Theorem.

Find the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$

Ans. Here $N = L = 4$

we know that

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi n k}{N}}$$

$$k = 0, 1, \dots, N-1$$

$$\begin{aligned} X(0) &= \sum_{n=0}^{4-1} x(n) e^{-j \frac{2\pi n \cdot 0}{4}} \\ &= \sum_{n=0}^3 x(n) e^0 = \sum_{n=0}^3 x(n) \cdot 1 \end{aligned}$$

$$\begin{aligned} &= x(0) + x(1) + x(2) + x(3) \\ &= 1 + 1 + 0 + 0 = 2 \end{aligned}$$

$$X(1) = \sum_{n=0}^3 x(n) e^{-j \frac{2\pi n \cdot 1}{4}} = \sum_{n=0}^3 x(n) e^{-j \frac{\pi n}{2}}$$

$$= x(0) e^{-j \frac{\pi \cdot 0}{2}} + x(1) e^{-j \frac{\pi \cdot 1}{2}} + x(2) e^{-j \frac{\pi \cdot 2}{2}} + x(3) e^{-j \frac{\pi \cdot 3}{2}}$$

$$= 1 \cdot e^0 + 1 \cdot e^{-j \pi/2} + 0 \cdot e^{-j \pi} + 0 \cdot e^{-j 3\pi/2}$$

$$= 1 + e^{-j \pi/2} + 0 + 0$$

$$= 1 + \cos \pi/2 - j \sin \pi/2$$

$$= 1 - 0 - j = 1 - j$$

$$= 1 - j$$

$$X(2) = \sum_{n=0}^3 x(n) e^{-j \frac{2\pi n \cdot 2}{4}} = \sum_{n=0}^3 x(n) e^{-j \pi n}$$

$$= x(0) e^{-j \pi \cdot 0} + x(1) e^{-j \pi \cdot 1} + x(2) e^{-j \pi \cdot 2} + x(3) e^{-j \pi \cdot 3}$$

$$= 1 \cdot e^0 + 1 \cdot e^{-j \pi} + 0 + 0$$

$$= 1 + e^{-j \pi} = 1 + \cos \pi - j \sin \pi$$

$$= 1 + (-1) - j \cdot 0 = 1 - 1 = 0$$

$$X(3) = \sum_{n=0}^3 x(n) e^{-j \frac{2\pi n 3}{4}} = \sum_{n=0}^3 x(n) e^{-j \frac{3\pi n}{2}}$$

$$= x(0) e^{-j0} + x(1) e^{-j \frac{3\pi}{2}} + 0 + 0$$

$$= 1 \cdot 1 + 1 e^{-j \frac{3\pi}{2}}$$

$$= 1 + \cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2}$$

$$= 1 + 0 - j(-1)$$

$$= 1 + j$$

$$\text{So } X(k) = \{ 2, 1-j, 0, 1+j \}$$

Laplace Transform for continuous time signals & system

Definition of Complex Frequency:

The complex frequency is represented as

$$s = \sigma + j\omega$$

When σ = Neper frequency in neper per second.

ω = Radian (or Real) frequency in radian per second -
(Neper frequency is a measure of how quickly a ckt's

transient response decays over time. It is the real part of complex frequency & represents the exponential decay rate of a signal).

$$\text{Let } x(t) = A e^{st} = A e^{(\sigma + j\omega)t}$$

When $\sigma = 0$

$$\begin{aligned} x(t) &= A e^{st} = A e^{j\omega t} \\ &= A (\cos \omega t + j \sin \omega t) = A \cos \omega t + j A \sin \omega t \end{aligned}$$

Here the real part $A \cos \omega t$
imaginary part $A \sin \omega t$.

When $\omega = 0$

$$x(t) = A e^{st} = A e^{\sigma t}$$

Here if σ is positive, signal will be exponentially increasing.
if σ is negative, signal will be exponentially decaying.

Complex Frequency Plane (s-plane)

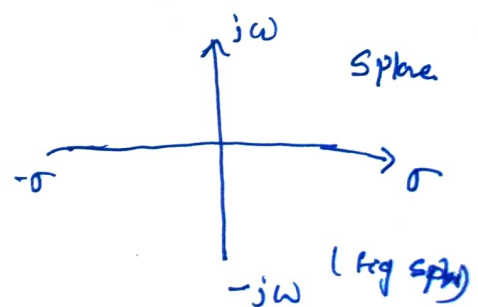
The complex frequency defined as, $s = \sigma + j\omega$

Where σ is the real part of s &

$j\omega$ is the imaginary part of s

Real part σ & imaginary part ω can take values from $-\infty$ to ∞ . They are represented on a

two dimensional complex plane along with horizontal axis & vertical axis. This is called complex freq. plane or s-plane.



Laplace Transform:

→ A time domain signal $x(t)$ can be transformed into S domain by multiplying the signal by e^{-st} and integrate from $-\infty$ to ∞ .

→ The transformed signal is represented by $X(s)$ & the transformation is denoted by $\mathcal{L}$.

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

For causal input signal $\mathcal{L}[x(t)] = X(s) = \int_0^{\infty} x(t) e^{-st} dt$

The S-domain signal $X(s)$ can be represented to time domain signal $x(t)$ by using inverse Laplace Transform

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi} \int_{\sigma-j\omega}^{\sigma+j\omega} X(s) e^{st} ds$$

Q. Determine Laplace transform of a signal $x(t) = e^{At} u(t)$

Ans. The Laplace transform of the signal can be determined as

$$\begin{aligned}
 X(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \\
 &= \int_{-\infty}^{\infty} e^{At} u(t) e^{-st} dt \\
 &= \int_0^{\infty} e^{At} e^{-st} dt
 \end{aligned}
 \quad \Bigg| \quad
 \begin{aligned}
 &= \int_0^{\infty} e^{-(A+s)t} dt = \frac{1}{-(A+s)} e^{-(A+s)t} \Big|_0^{\infty} \\
 &= \frac{1}{-(A+s)} (e^{-\infty} - e^0) \\
 &= \frac{1}{(-A+s)} (0 - 1) = \frac{1}{A+s}
 \end{aligned}$$

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$$\text{So } X(s) = \frac{1}{A+s}$$

Since $s = \sigma + j\omega$, by putting $\omega = 0$

$$X(s) = \frac{1}{A + \sigma + j\omega}$$

$$= \frac{1}{(\sigma + A) + j\omega}, \quad \sigma + A > 0$$

$$= \frac{1}{(\sigma + A) + j\omega}, \quad \sigma > -A$$

Hence

$$X(s) = \frac{1}{s + A}, \quad \sigma > -A$$

The Region of Convergence (ROC):

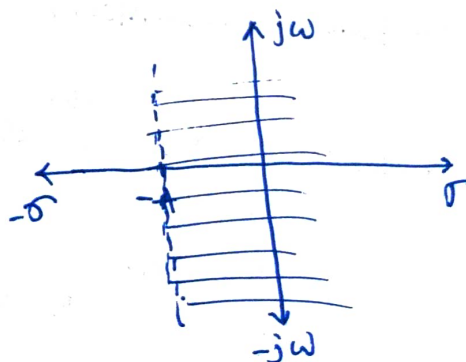
The range of values of s for which the integral of equation $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$ converges is called the region of convergence (ROC) of the Laplace transform.

→ The ROC can be displayed on s plane, which is a complex variable.

→ The horizontal & vertical axis are called the σ -axis & $j\omega$ axis.

Ex: $X(s) = \frac{1}{s + A}, \quad \sigma > -A$

ROC: $\sigma > -A$



1. Determine the Laplace transform of a continuous time signal

$$x(t) = 2e^{-3t}u(t) - e^{-2t}u(t)$$

Also find the ROC of Laplace transform of above signal.

Ans. the Laplace transform of $x(t)$ is determined as

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} [2e^{-3t}u(t) - e^{-2t}u(t)] e^{-st} dt$$

$$= \int_0^{\infty} 2e^{-3t}u(t) e^{-st} dt - \int_0^{\infty} e^{-2t}u(t) e^{-st} dt$$

$$= 2 \int_0^{\infty} e^{-3t} \cdot e^{-st} dt - \int_0^{\infty} e^{-2t} \cdot e^{-st} dt$$

$$= 2 \int_0^{\infty} e^{-(s+3)t} dt - \int_0^{\infty} e^{-(s+2)t} dt$$

$$= 2 \left[\frac{e^{-(s+3)t}}{-(s+3)} \right]_0^{\infty} - \left[\frac{e^{-(s+2)t}}{-(s+2)} \right]_0^{\infty}$$

$$= 2 \left[\frac{e^{-\infty} - e^0}{-(s+3)} \right] - \left[\frac{e^{-\infty} - e^0}{-(s+2)} \right]$$

$$= 2 \left[\frac{0-1}{-(s+3)} \right] - \left[\frac{0-1}{-(s+2)} \right] = 2 \left[\frac{-1}{-(s+3)} \right] - \left[\frac{-1}{-(s+2)} \right]$$

$$= \frac{2}{s+3} - \frac{1}{s+2}$$

$$2e^{-3t} u(t) \xleftrightarrow{LT} \frac{2}{s+3}$$

$$R_1: \sigma > -3$$

$$e^{-2t} u(t) \xleftrightarrow{LT} \frac{1}{s+2}$$

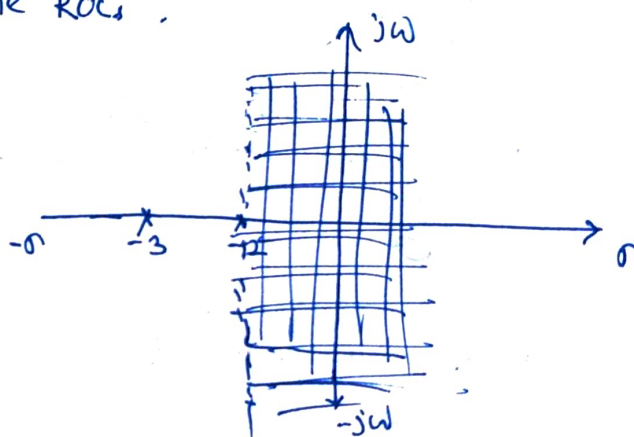
$$R_2: \sigma > -2$$

$$\text{Then } 2e^{-3t} u(t) - e^{-2t} u(t) \longleftrightarrow \frac{2}{(s+3)} - \frac{1}{(s+2)} = \frac{2(s+2) - (s+3)}{(s+3)(s+2)}$$

$$= \frac{2s+4-s-3}{(s+3)(s+2)} = \frac{(s-1)}{(s+3)(s+2)}$$

$$R = R_1 \cap R_2 = \sigma > -2$$

Where R_1, R_2 & R are the ROCs.



Properties of ROC:

1. The ROC of the Laplace transform of signal $x(t)$ consists of those values of s for which $x(t)e^{\sigma t}$ is absolutely integrable. i.e.

$$\int_{-\infty}^{\infty} x(t) e^{\sigma t} dt < \infty$$

2. The ROC for rational Laplace transforms does not contain any pole.

3. If $x(t)$ is a finite-duration signal & is absolutely integrable, then ROC of Laplace transform $x(t)$ is the entire s plane.

4. If $x(t)$ is a right-sided signal & if ~~now~~ the line $\text{Real}\{s\} = \sigma_0$ is in the Roc, then all the values of s for which $\text{Real}\{s\} > \sigma_0$ will also be in Roc.

5. If $x(t)$ is a left-sided signal & if the line $\text{Real}\{s\} = \sigma_0$ is in the Roc, then all the values of s for which $\text{Real}\{s\} < \sigma_0$ will also be in Roc.

6. If $x(t)$ is a two sided signal, and if the line $\text{Real}\{s\} = \sigma_0$ is in the Roc, then the Roc of the Laplace transform of $x(t)$ will consist of a strip in the s -plane which includes the line $\text{Real}\{s\} = \sigma_0$.

7. If the Laplace transform $X(s)$ of signal $x(t)$ is a rational function & then its Roc is bounded by poles or extends to infinity.

8. If the Laplace transform $X(s)$ of the signal $x(t)$ is a rational function, then if signal $x(t)$ is right-sided, the Roc is the region in the s -plane to the right of the right most pole.

Case-I: Right sided signal

① Find Laplace transform & Roc of the signal

$$x(t) = e^{-bt} u(t) \quad \text{for } b > 0$$

Ans. Laplace transform of the signal $x(t)$ is

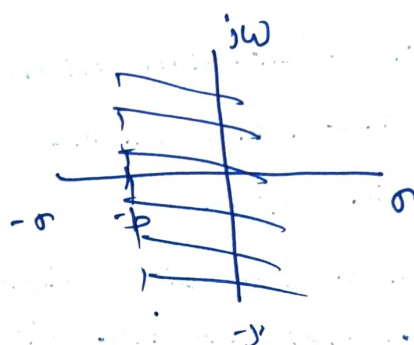
$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_0^{\infty} e^{-bt} e^{-st} dt = \int_0^{\infty} e^{-(s+b)t} dt \\ &= \left[\frac{e^{-(s+b)t}}{-(s+b)} \right]_0^{\infty} = \frac{e^{-\infty} - e^0}{-(s+b)} = \frac{0-1}{-(s+b)} = \frac{1}{s+b} \end{aligned}$$

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The $x(s)$ converges for $\sigma > -b$, where $s = \sigma + j\omega$

Therefore for causal signal ROC is the right side



Case-II (Left sided signal)

① $x(t) = e^{-bt} u(t)$ where $b > 0$, for $t \leq 0$ find LT & ROC.

Ans

$$\begin{aligned}
 x(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \\
 &= \int_{-\infty}^{\infty} e^{-bt} u(t) e^{-st} dt \\
 &= \int_{-\infty}^0 e^{-bt} e^{-st} dt \quad (\because t \leq 0) \\
 &= \int_{-\infty}^0 e^{-(s+b)t} dt = \left[\frac{e^{-(s+b)t}}{-(s+b)} \right]_{-\infty}^0 \\
 &= \frac{1}{s+b}
 \end{aligned}$$

$$\Rightarrow X(s) = \int_{-\infty}^0 \frac{e^{-(s+b)t}}{-(s+b)} dt$$

$$= -\frac{1}{(s+b)} \left[e^{-0} - e^{(s+b)\infty} \right]$$

$$= -\frac{1}{(s+b)} + e^{(\sigma+j\omega+b)\infty}$$

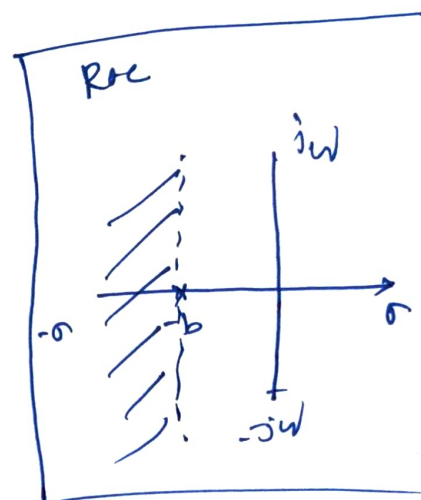
if $\sigma+b > 0 \Rightarrow \sigma > -b$, $e^{\infty} = \infty$

if $\sigma+b < 0 \Rightarrow \sigma < -b$, $e^{\infty} = 0$

Hence $X(s)$ converges when $\sigma < -b$

$$\therefore X(s) = -\frac{1}{s+b} + \frac{0}{s+b} = -\frac{1}{s+b}$$

Therefore for an anticausal signal Roc is on the left of pole $\sigma = -b$.



$$X(s) = \left[\frac{e^{-(s+b)t}}{-(s+b)} \right]_{-\infty}^0$$

To solve the integral, we evaluate it at the limits

$$\begin{aligned} X(s) &= \frac{e^{-(s+b) \cdot 0}}{-(s+b)} - \lim_{t \rightarrow -\infty} \frac{e^{-(s+b)t}}{-(s+b)} \\ &= \frac{1}{-(s+b)} - \lim_{t \rightarrow -\infty} \frac{e^{-(s+b)t}}{-(s+b)} \end{aligned}$$

For limit to converge, the ~~exponent~~ exponent $-(s+b)t$ must go to $-\infty$ as $t \rightarrow -\infty$. This requires the real part of $(s+b)$ to be negative.

$s = \sigma + j\omega$ & b be a real num.

$$\begin{aligned} \text{So } \sigma + b < 0 &\Rightarrow \sigma < -b \\ &\Rightarrow s < -b \end{aligned}$$

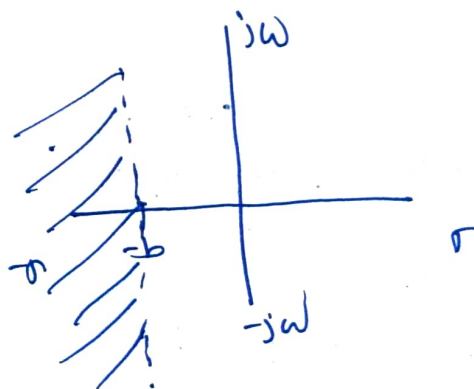
Under this condition the limit term becomes 0

$$\lim_{t \rightarrow -\infty} \frac{e^{-(s+b)t}}{-(s+b)} = 0$$

So Laplace transform is $X(s) = \frac{1}{-(s+b)} - 0 = -\frac{1}{s+b}$.

Roc : $s = \sigma + j\omega$

$$\begin{aligned} s+b &< 0 \\ \Rightarrow s &< -b \end{aligned}$$



Case III: Two sided signal

1. $x(t) = e^{-at} u(t) + e^{-bt} u(-t)$ where $a, b > 0$ & $a > b$
Find LT & its ROC

Ans: Using Laplace transfm $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$$\Rightarrow X(s) = \int_{-\infty}^{\infty} [e^{-at} u(t) + e^{-bt} u(-t)] e^{-st} dt$$

$$= \int_0^{\infty} e^{-at} u(t) e^{-st} dt + \int_{-\infty}^0 e^{-bt} u(-t) e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt + \int_{-\infty}^0 e^{-(s+b)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} + \left[\frac{e^{-(s+b)t}}{-(s+b)} \right]_{-\infty}^0$$

Here when $(s+a) > 0 \Rightarrow s > -a$, $e^{\infty} = 0$

$(s+a) < 0 \Rightarrow s < -a$, $e^{\infty} = \infty$

$(s+b) > 0 \Rightarrow s > -b$, $e^{\infty} = \infty$

$(s+b) < 0 \Rightarrow s < -b$, $e^{\infty} = 0$

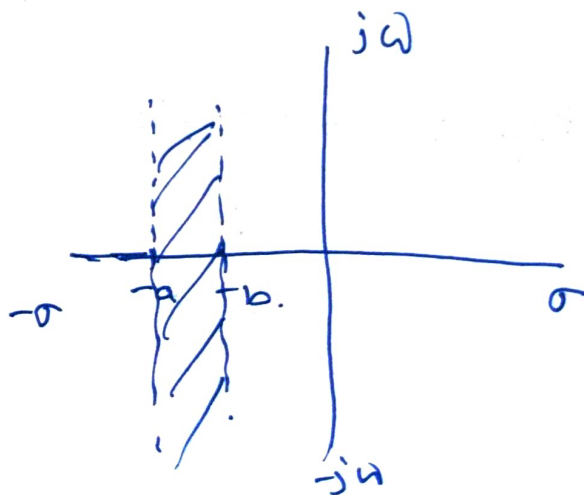
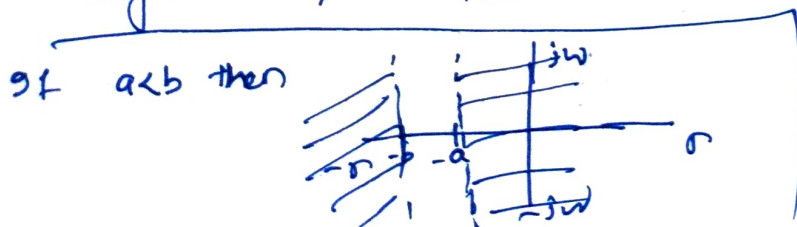
$$= \cancel{\left(-\frac{1}{s+a} \right)} \left[e^{\infty} - e^0 \right] + \left(-\frac{1}{s+b} \right) \left[e^0 - e^{\infty} \right]$$

$$= -\left(\frac{1}{s+a} \right) [0 - 1] + \left\{ -\frac{1}{s+b} \right\} [1 - 0] \quad (\because \text{we put } e^{\infty} = 0)$$

$$X(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$\begin{array}{l|l} s+a > 0 & s+b < 0 \\ \Rightarrow s > -a & \Rightarrow s < -b \end{array}$$

again $a > b$ so



Poles & Zeros: of System Function & Signals:

The Laplace transform is represented in terms of rational, i.e. it is a ratio of polynomials in the complex variable s .

$$X(s) = \frac{N(s)}{D(s)}$$

where N & D are numerator & denominators respectively.

* The roots of $N(s)$ are known as the zeros. For these values of s , $X(s)$ is zero.

* The roots of $D(s)$ are known as poles. For these values of s , $X(s)$ is infinite (∞).

→ The Region of convergence for the Laplace transform cannot contain any poles, because the corresponding integral is infinite.

* The graphical representation of $X(s)$ through its pole and zeros in s plane is referred to as the pole-zero plot of $X(s)$.

System Function:

The system function is written as

$$H(s) = \frac{Y(s)}{X(s)}$$

The transfer function of a continuous time system is defined as the ratio of Laplace transform of output and Laplace transform of input.

Transfer function of LTI continuous time system:

In general input-output relation of a LTI continuous time system is represented by constant coefficient differential eqⁿ as follows

$$\frac{d^n}{dt^n} y(t) + p_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + p_2 \frac{d^{n-2}}{dt^{n-2}} y(t) + \dots + p_{n-1} \frac{d}{dt} y(t) + p_n y(t) =$$
$$z_0 \frac{d^m}{dt^m} x(t) + z_1 \frac{d^{m-1}}{dt^{m-1}} x(t) + z_2 \frac{d^{m-2}}{dt^{m-2}} x(t) + \dots + z_{m-1} \frac{d}{dt} x(t) + z_m x(t)$$

where n is order of system and $m \leq n$

On taking LT above eqⁿ & taking zero initial condition

$$\boxed{\frac{Y(s)}{X(s)} = \frac{z_0 s^m + z_1 s^{m-1} + z_2 s^{m-2} + \dots + z_{m-1} s + z_m}{p_0 s^n + p_1 s^{n-1} + p_2 s^{n-2} + \dots + p_{n-1} s + p_n}}$$

Above function is the transfer function of an LTI system.

Impulse response & transfer function:

Let $x(t)$ = input to LTI system, $y(t)$ = output for input $x(t)$

$h(t)$ = impulse response.

The response $y(t)$ of the continuous time system is given by convolution of input & impulse response. So

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

Taking Laplace transform

$$Y(s) = X(s) \cdot H(s)$$

$$\Rightarrow \boxed{H(s) = \frac{Y(s)}{X(s)} = K \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}}$$

The transfer function of LTI system continuous time system is also given by Laplace transform of the impulse response.

1. Draw the pole-zero plot.

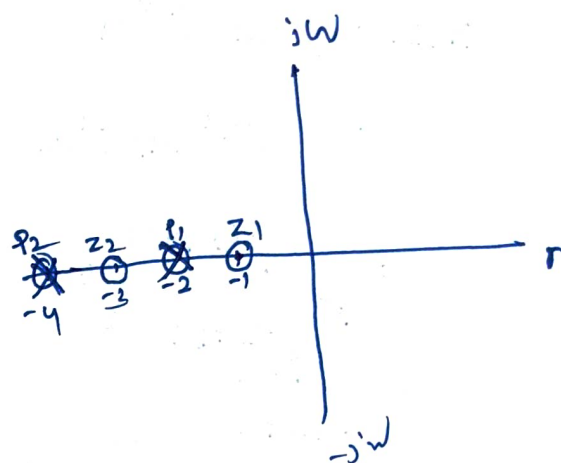
$$Y(s) = 4 \frac{(s+1)(s+3)}{(s+2)(s+4)}$$

Ans. Here zeros occur at

$$\begin{aligned} s+1 &= 0, & s+3 &= 0 \\ \Rightarrow s &= -1 & \Rightarrow s &= -3. \end{aligned}$$

Here poles occur at

$$\begin{aligned} s+2 &= 0 \Rightarrow s = -2 \\ s+4 &= 0 \Rightarrow s = -4. \end{aligned}$$



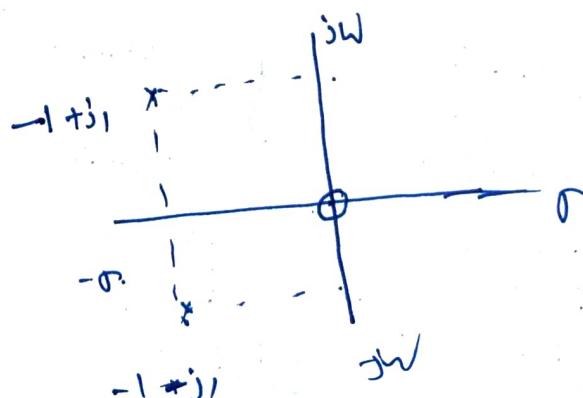
Q. Draw Pole-zero plot.

$$H(s) = \frac{10s}{s^2 + 2s + 2}$$

$$\text{Ans. } H(s) = \frac{10s}{s^2 + 2s + 2} = \frac{10s}{(s+1-j1)(s+1+j1)}$$

Zero occurs at $s = 0$

Poles occur at $s+1-j1 = 0$ and $s+1+j1 = 0$
 $\Rightarrow s = -1+j1$ and $\Rightarrow s = -1-j1$



Q. Find the transfer function of LTI system with input $x(t) = e^{4t} u(t)$ & output $y(t) = [e^{2t} - e^{3t}] u(t)$.

Ans: The transfer function is given by $H(s) = \frac{Y(s)}{X(s)}$

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} e^{4t} u(t) e^{-st} dt \\ &= \int_0^{\infty} e^{-(s+4)t} dt \\ &= \left[\frac{e^{-(s+4)t}}{-(s+4)} \right]_0^{\infty} \\ &= \frac{1}{-(s+4)} [e^{-\infty} - e^0] \end{aligned}$$

$$\Rightarrow X(s) = \frac{1}{s+4}$$

Roc: $\sigma < -4$

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} [e^{2t} - e^{3t}] u(t) e^{-st} dt \\ &= \int_0^{\infty} e^{-2t} e^{-st} dt - \int_0^{\infty} e^{-3t} e^{-st} dt \\ &= \int_0^{\infty} e^{-(s+2)t} dt - \int_0^{\infty} e^{-(s+3)t} dt \\ &= \left[\frac{e^{-(s+2)t}}{-(s+2)} \right]_0^{\infty} - \left[\frac{e^{-(s+3)t}}{-(s+3)} \right]_0^{\infty} \\ &= \frac{e^{-\infty} - e^0}{-(s+2)} - \left[\frac{e^{-\infty} - e^0}{-(s+3)} \right] \\ &= \frac{-1}{-(s+2)} - \left[\frac{-1}{-(s+3)} \right] \end{aligned}$$

$$\Rightarrow Y(s) = \frac{1}{s+2} - \frac{1}{s+3}$$

$$= \frac{s+3 - s-2}{(s+2)(s+3)} = \frac{1}{(s+2)(s+3)}$$

Roc: $\sigma < -2 \cap \sigma < -3$

$$\text{So } Y(s) = \frac{1}{(s+2)(s+3)}$$

$$\begin{aligned} H(s) &= \frac{Y(s)}{X(s)} \\ &= \frac{1}{(s+2)(s+3)} \cdot \frac{1}{\frac{1}{s+4}} \end{aligned}$$

$$\Rightarrow H(s) = \frac{(s+4)}{(s+2)(s+3)}$$

Structure Realization:

→ The block diagram representation of a difference equation is called realization structure.

→ This diagram indicates the manner in which the computations are performed.

→ The different types of structure realization are

- i) Direct Form I
- ii) Direct Form II
- iii) cascade form
- iv) Parallel form.

Direct Form I : It is the direct implementation of difference equation or transfer function describing the system.

- It uses separate integrators for input and output variables.
- It provides direct relation betⁿ time domain & s domain equation.

a. Realize the system with direct form I.

$$H(s) = \frac{s+s}{s^2+2s+4}$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{s+s}{s^2+2s+4}$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{\frac{s^2}{s^2} + \frac{s}{s^2}}{\frac{s^2}{s^2} + \frac{2s}{s^2} + \frac{4}{s^2}} = \frac{\bar{s}^1 + s\bar{s}^2}{1 + 2\bar{s}^1 + 4\bar{s}^2}$$

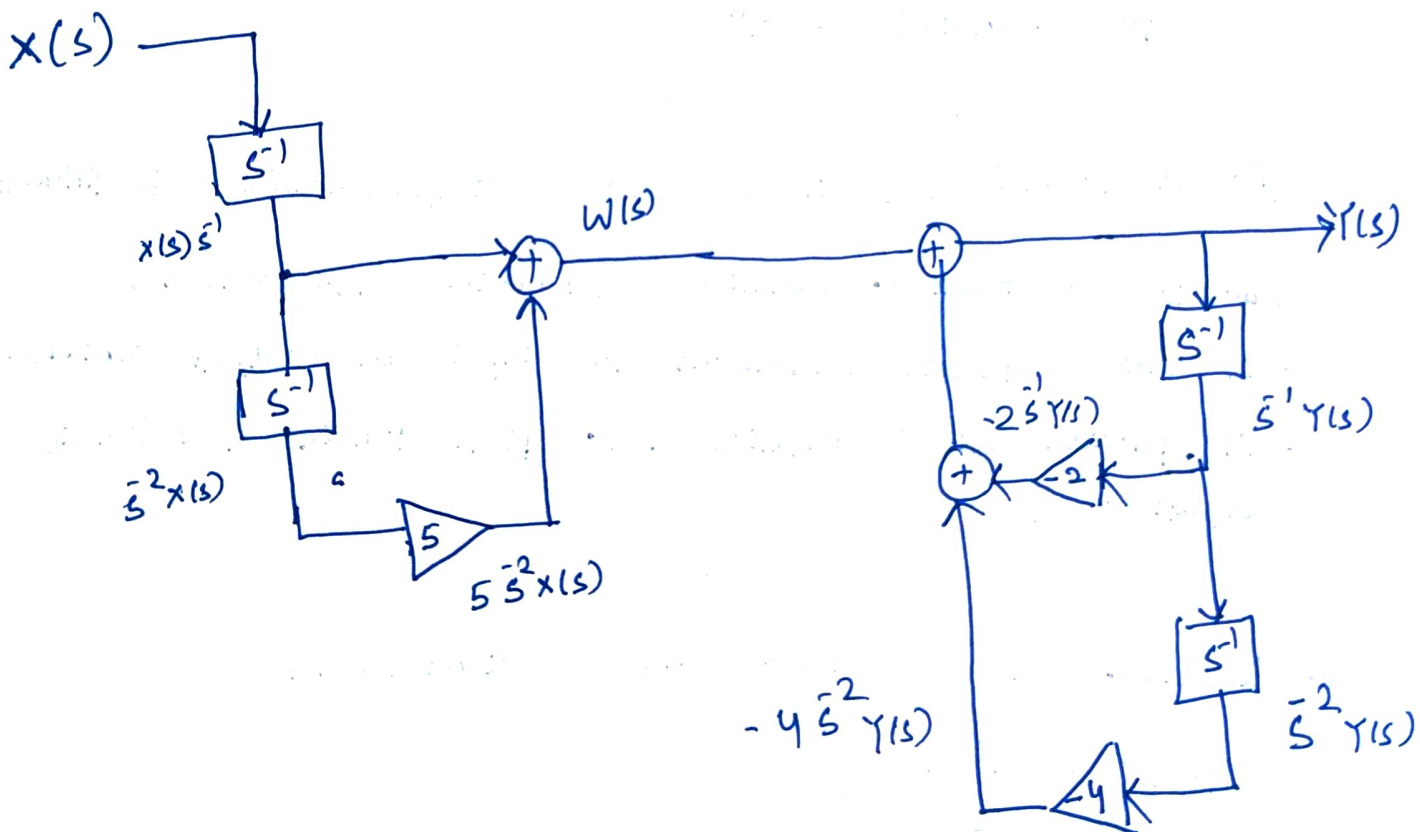
$$\Rightarrow Y(s) [1 + 2s^{-1} + 4s^{-2}] = X(s) [s^{-1} + 5s^{-2}]$$

$$\Rightarrow Y(s) + 2s^{-1}Y(s) + 4s^{-2}Y(s) = s^{-1}X(s) + 5s^{-2}X(s)$$

$$\text{Let } W(s) = s^{-1}X(s) + 5s^{-2}X(s)$$

$$\Rightarrow Y(s) = W(s) - 2s^{-1}Y(s) - 4s^{-2}Y(s)$$

$$\begin{aligned} \Rightarrow Y(s) &= s^{-1}X(s) + 5s^{-2}X(s) - 2s^{-1}Y(s) - 4s^{-2}Y(s) \\ &= W(s) - 2s^{-1}Y(s) - 4s^{-2}Y(s) \end{aligned}$$



Direct form-II : It is the direct implementation of differential equation or transfer function describe the system.

→ Instead of using separate integrators for integrating input & output variable separately, an intermediate variable is integrated.

- It provides direct relation betⁿ time domain & s domain eqs.

Q. Realize using direct form II $H(s) = \frac{s+4}{s^2+3s+5}$

Ans. $H(s) = \frac{Y(s)}{X(s)} = \frac{s+4}{s^2+3s+5}$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{\frac{s}{s^2} + \frac{4}{s^2}}{\frac{s^2}{s^2} + \frac{3s}{s^2} + \frac{5}{s^2}}$$

($\because$ Transfer function expressed in -ve power of s)

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{s^{-1} + 4s^{-2}}{1 + 3s^{-1} + 5s^{-2}}$$

Dividing transfer function into two parts we have

$$\begin{aligned} H(s) = \frac{Y(s)}{X(s)} &= \frac{Y(s)}{W(s)} \cdot \frac{W(s)}{X(s)} \\ &= \frac{Y(s)}{W(s)} \cdot \frac{W(s)}{X(s)} \\ &= (s^{-1} + 4s^{-2}) \cdot \frac{1}{1 + 3s^{-1} + 5s^{-2}} \end{aligned}$$

When $\frac{Y(s)}{W(s)} = s^{-1} + 4s^{-2}$

& $\frac{W(s)}{X(s)} = \frac{1}{1 + 3s^{-1} + 5s^{-2}}$

$$\frac{Y(s)}{W(s)} = s^{-1} + 4s^{-2}$$

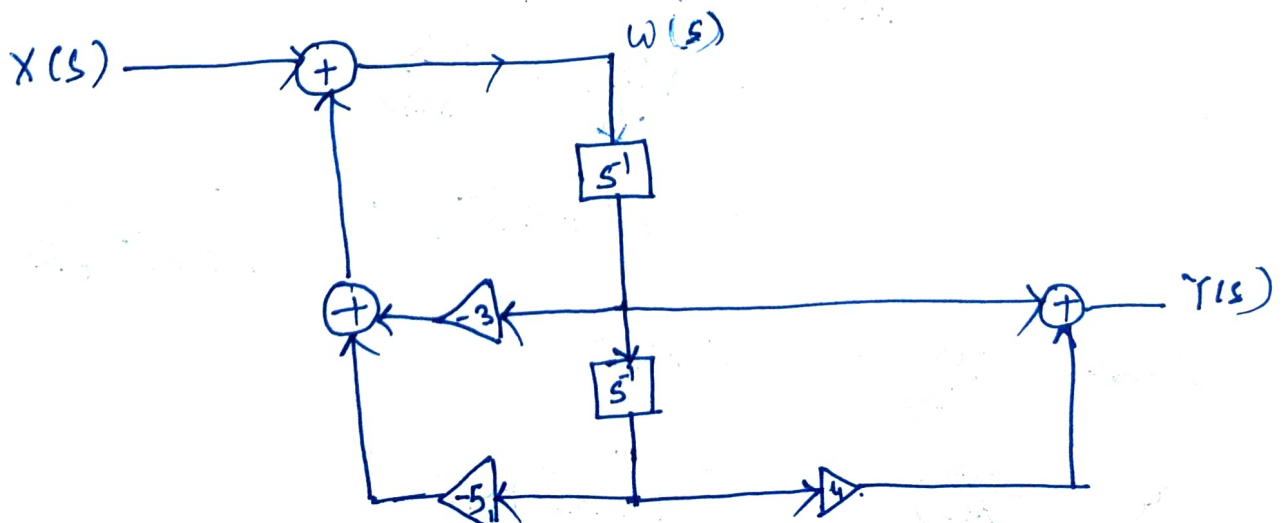
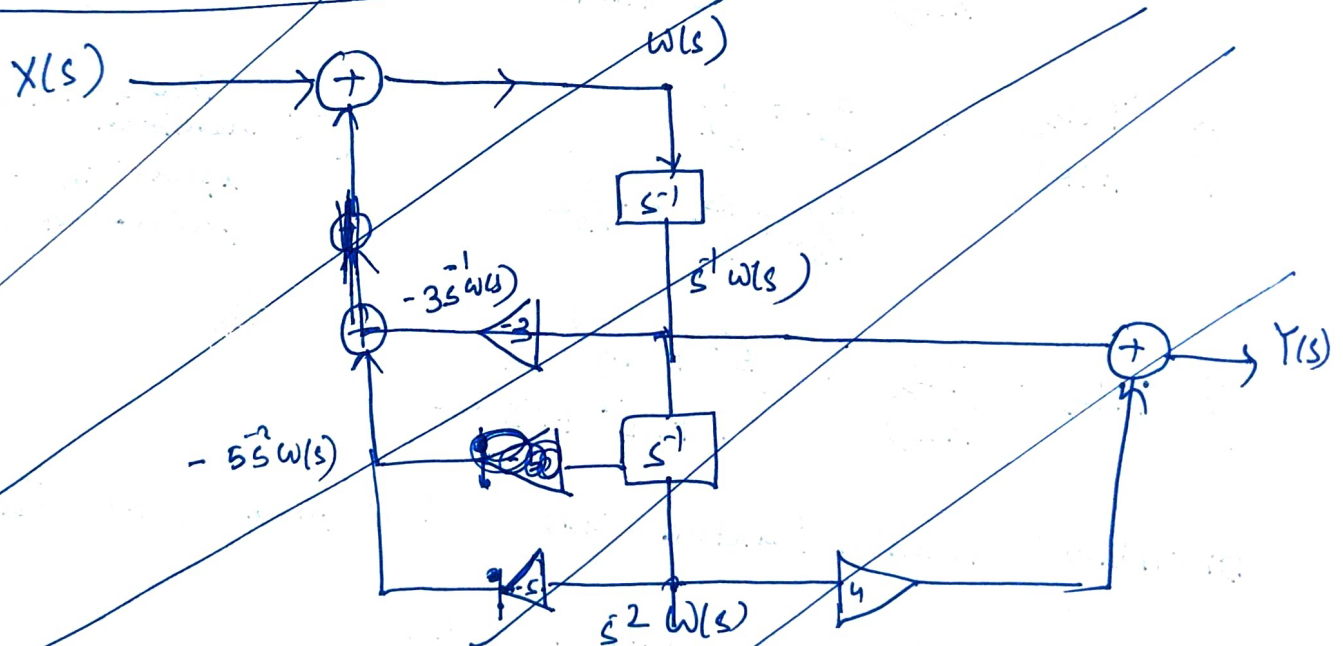
$$\Rightarrow Y(s) = s^{-1} W(s) + 4s^{-2} W(s)$$

Agun $\frac{W(s)}{X(s)} = \frac{1}{1+3s^{-1}+5s^{-2}}$

$$\Rightarrow X(s) = W(s) [1 + 3s^{-1} + 5s^{-2}]$$

$$\Rightarrow X(s) = W(s) + 3s^{-1} W(s) + 5s^{-2} W(s)$$

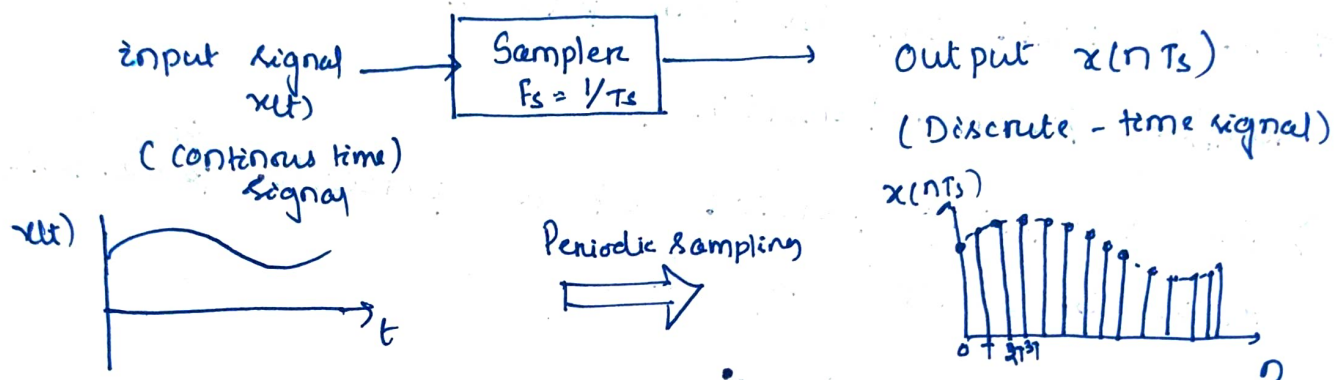
$$\Rightarrow W(s) = X(s) - 3s^{-1} W(s) - 5s^{-2} W(s)$$



Sampling Theorem & its implication:

* Sampling is a process of converting a continuous time signal into a discrete time signal.

→ By periodic sampling, a continuous-time signal $x(t)$ converted into a discrete time signal $x(nT_s)$, where T_s is the sampling period.



- The time interval between successive samples is T_s & sampling frequency is given by $F_s = \frac{1}{T_s}$ Hz.

Sampling Theorem: Sampling theorem states that: For perfect reconstruction of continuous time signal from its sample, the sampling rate or sampling frequency should be greater than or equal to twice the highest frequency present in the signal. i.e.

$$F_s \geq 2F_m$$

Nyquist rate: It is defined as minimum sampling rate required for perfect reconstruction of continuous time signal from its samples.

Nyquist rate = $2 \times$ highest frequency component of message signal.

$$= 2 \times F_m$$

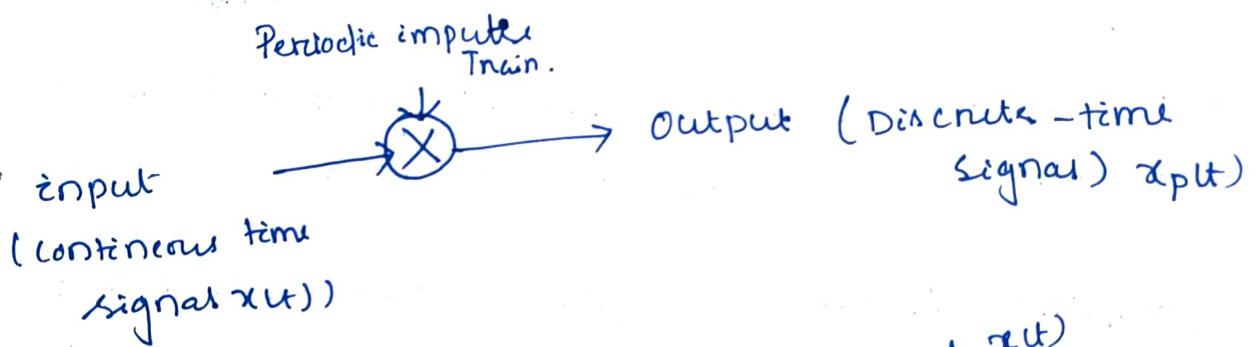
→ Sampling theorem can be developed by sampling a continuous-time signal at regular intervals.

→ A Periodic impulse train multiplied by the continuous-time signal $x(t)$.

→ This mechanism is called impulse train sampling.

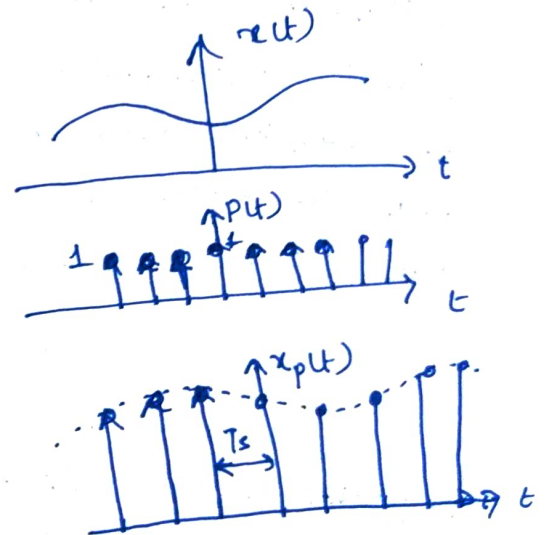
→ The periodic impulse train $p(t)$ is also called the sampling ~~function~~ function.

→ The sampling period of a periodic ~~periodic~~ impulse train is denoted by T_s .



→ The reciprocal of sampling period T_s is called sampling frequency f_s .

→ The fundamental frequency of periodic impulse train $p(t)$ is $\omega_s = 2\pi f_s$



In time domain $x_p(t) = x(t) \cdot p(t)$

& $p(t)$ is given by $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

By sampling property $x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$

By applying property $x_p(t)$ is an impulse train with amplitude of impulse equal to the sample of $x(t)$ at interval spanned by T_s .

By the multiplication property of Fourier transform

$$x_p(t) = x(t) \cdot p(t) \xrightarrow{F.T} X_p(j\omega) = \frac{1}{2\pi} [X(j\omega) * P(j\omega)]$$

$$FT[p(t)] = FT\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$

$$= \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

By substituting

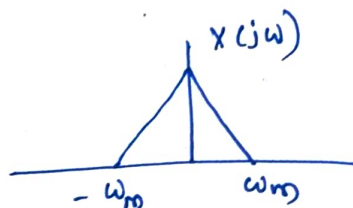
$$So X_p(j\omega) = \frac{1}{2\pi} [X(j\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)]$$

$$\Rightarrow X_p(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$\because X(j\omega) * \delta(\omega - \omega_0) = X[j(\omega - \omega_0)]$$

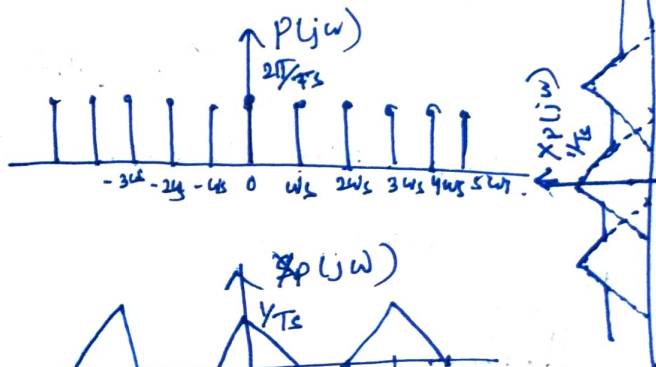
Hence $X_p(j\omega)$ is a function of frequency ω consisting of a superposition of shifted version (on shifted replicas) of $X(j\omega)$ & scaled by $1/T_s$.

(1)



(Spectrum of original signal.)

(2)

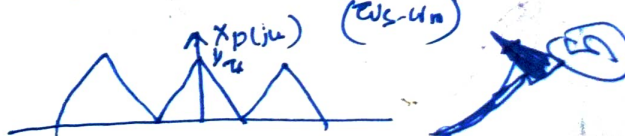


(3)

($\omega_s > 2\omega_m$)

(4)

($\omega_s = 2\omega_m$)



There are three cases:

- i) If $\omega_s > 2\omega_m$ & thus there is no overlap between shifted replicas.
- ii) If $\omega_s = 2\omega_m$, then the spectrum of $x_p(t)$ i.e. $X_p(j\omega)$ replicas ~~are~~ not ^{overlap} ~~overlapping~~.
- iii) If $\omega_s < 2\omega_m$, there is an overlap between the shifted replicas.

* Thus $\omega_s \geq 2\omega_m$ required to reconstruct the original signal $x(t)$.

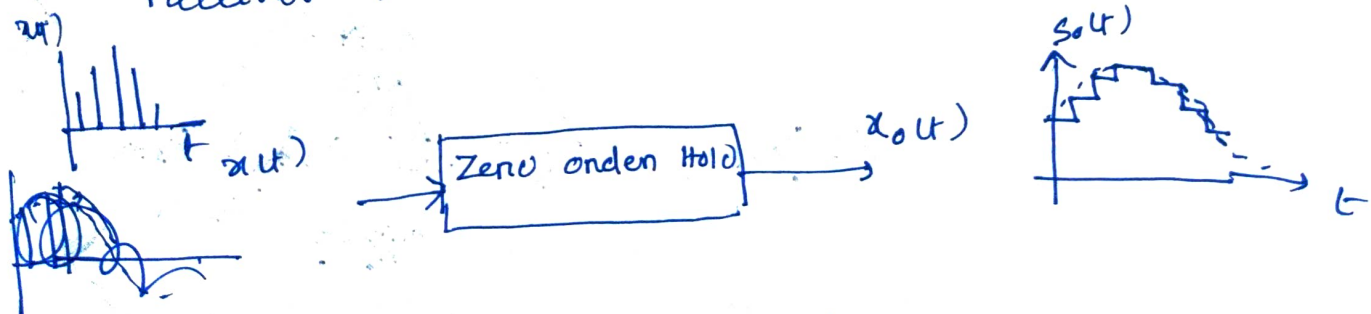
Reconstruction of Analog signal:

Continuous signal or Analog signal can be reconstructed from digital signal by following methods.

1. Zero order hold
2. First order hold

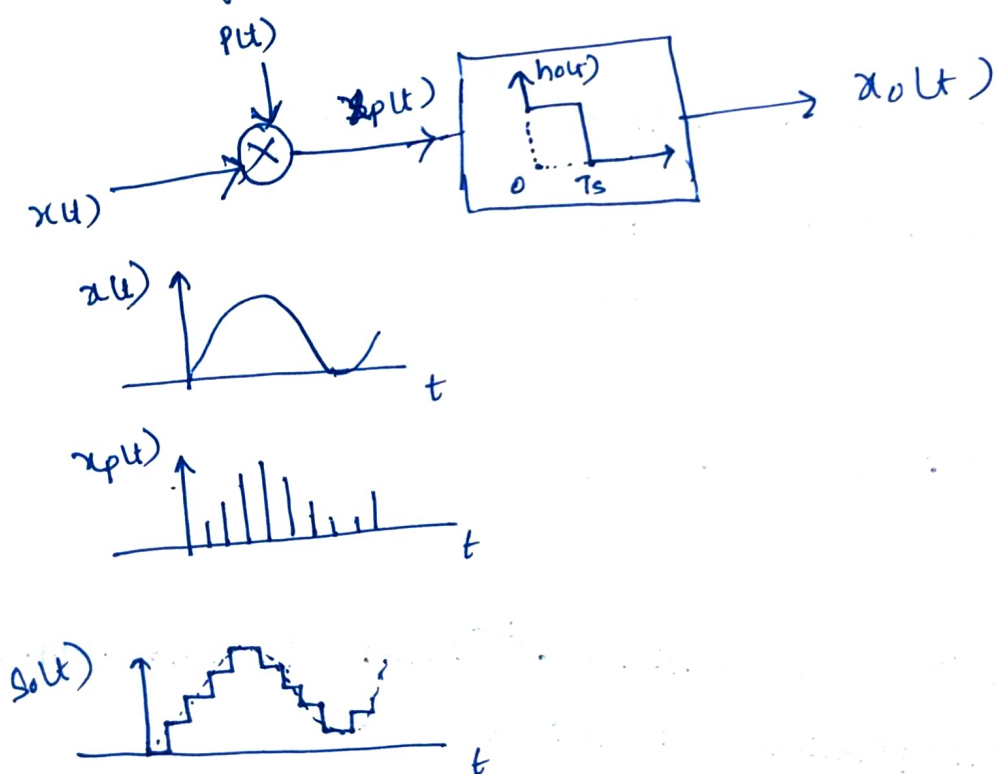
Zero Order Hold:

In this technique a given sample is held for sample interval until next sample is received.



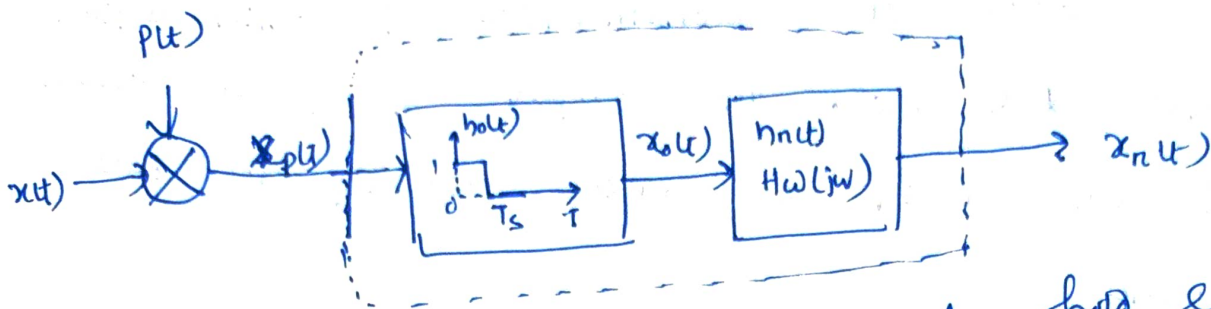
→ The recovery or reconstruction of continuous time signal $x(t)$ from the output of a zero-order hold can again be carried out by low pass filtering.

→ The output $x_0(t)$ of the zero order hold can be generated in principle by impulse train sampling followed by a continuous time LTI system with rectangular impulse response.



* For reconstruction of $x(t)$ from zero order hold $x_0(t)$, the processing of $x_0(t)$ with continuous LTI system with impulse response $h(t)$.
 → The frequency response is denoted by ~~$H(j\omega)$~~
 $H_r(j\omega)$.

→ Cascade of zero order hold system with reconstruction low pass filter shown in next fig.



(cascade combination of zero order hold system with reconstruction filter)

→ Here $x_n(t) = x(t)$ ~~if~~ if cascade combination of $h_0(t)$ & $h_n(t)$ are ideal low pass filters

The impulse response $h_0(t)$ is defined as

$$h_0(t) = \begin{cases} 1 & , 0 \leq t \leq T_s \\ 0 & , \text{otherwise} \end{cases}$$

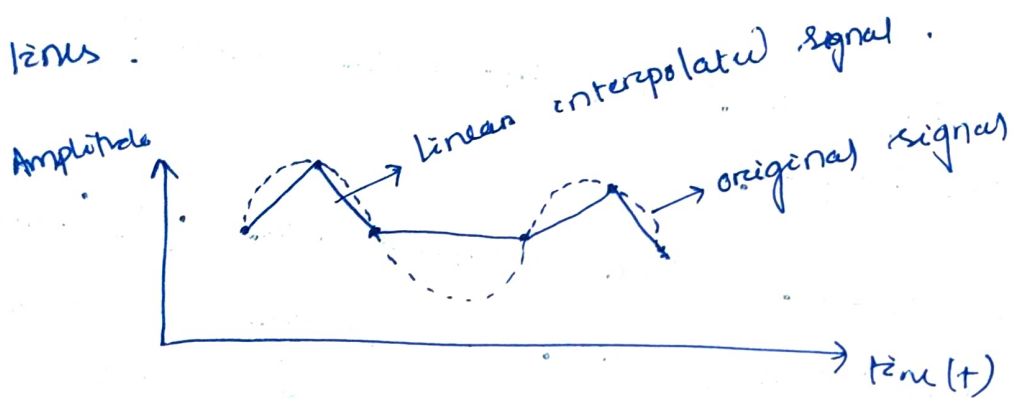
$$\& H_{\text{recon}}(j\omega) = H_0(j\omega) \cdot H_n(j\omega)$$

Reconstruction of continuous-time signal from its samples using interpolation techniques -

→ Interpolation is ~~one~~ the commonly used procedure for reconstructing a continuous time signal from its sample values.

→ Interpolation gives either approximate or exact reconstruction of signal.

- One simple interpolation is zero order hold.
- Another useful form of interpolation is the linear interpolation.
- In linear interpolation, the adjacent samples (or sample points) are connected by straight lines.



- For a bandlimited continuous-time signal, if the sample instants are sufficiently close, then, the signal can be reconstructed exactly by using low pass filter.
- In this case an exact interpolation can be carried out between various sample points.
- The output of the reconstruction low pass filter given as

$$x_{re}(t) = x_p(t) * h(t)$$

Where $h(t)$ is impulse response.

Aliasing Phenomenon

- When sampling frequency is less than the Nyquist rate it is called Under sampling.
- If sampling rate or frequency is less than Nyquist rate aliasing phenomenon occurs.
- If sampling rate $>$ Nyquist rate then it is called over sampling.
- Aliasing occurs when sampling rate ($F_s < 2f_m$) is not sufficiently high to prevent the shifting of high frequency information into lower frequencies.
- Aliasing phenomenon is defined as a phenomenon of high frequency components in a spectrum of a continuous time signal seemingly taking on the identity of a lower frequency in spectrum of its sampled version.
- Corrective measures which are used to eliminate aliasing.
 - 1) Pre-aliasing low pass filter is used before sampling.
 - 2) Pre aliasing low pass filtered signal is sampled at a rate $F_s > 2f_m$.

Need of filter: in signal & system:

① Filters are used in audio system, communication, Radar & sonar, image & video processing instrumentation & control system. etc.