Digital Signal Processing

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Advantages of DSP & ASP:

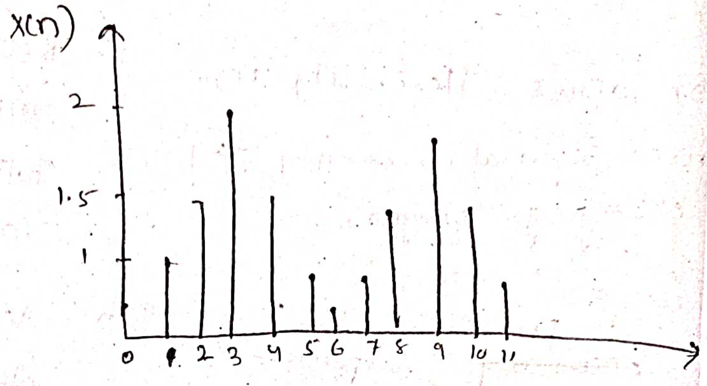
DSP	ASP
1. It allows flexibility in reconfiguration simply by changing program.	1. Reconfiguration of analog system implies a redesign of hardware followed by testing & varification of system.
2. It provide ~~Accuracy~~ superior control of accuracy.	2. Accuracy of ~~analog~~ ASP system is difficult due to tolerance.
3. ~~Easily~~ digital signal easily store on magnetic media without loss of fidelity.	3. Analog signals are not easily stored.
4. Low cost	4. High cost.
5. ~~It can~~ Mathematical operation may apply on digital computer using software.	5. Difficult to apply mathematical operation.

Digital Time Signal:

- A discrete - time signal x(n) is a function of an ~~integer~~ independent variable that is an integer.
- A discrete time signal is not defined at instants between two successive samples.
- Discrete time signals are defined only at certain value of time.
- DTS can be represented mathematically by a sequence of real or complex numbers.
- DTS is discrete in time but continous in amplitude.

Signal Representation:

1. Graphical Representation:



2. Functional Representation:

$x(n) = 2$ $\quad n = 3, 9$

$= 1.5$ $\quad n = 2, 4, 8, 10$

$= 1$ $\quad n = 1, 5, 7, 11$

$= .5$ $\quad n = 6$

3. Tabular Representation:

n	0	1	2	3	4	5	6	7	8	9	10	11
x(n)	1	1	1.5	2	1.5	1	.5					

4. Sequencial Representation:

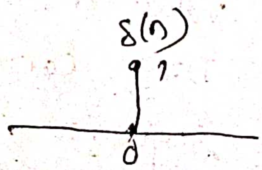
$$x(n) = \{\underset{\uparrow}{1}, 1, 1.5, 2, 1.5, 1, .5, 1.5, ..\}$$

Elementary Discrete time Signal:

1. Unit Sample Sequence: It is denoted as $\delta(n)$.

It is defined as:

$$\delta(n) = 1 \text{ for } n = 0$$
$$= 0 \text{ for } n \neq 0$$

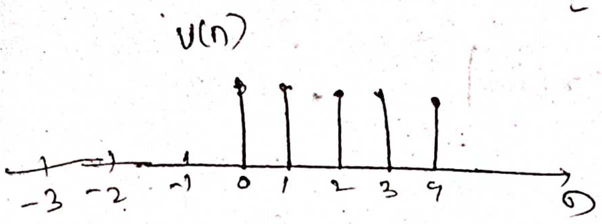


- Also known as unit impulse.

Unit Step Sequence.

- It is denoted as $u(n)$
- It is defined as

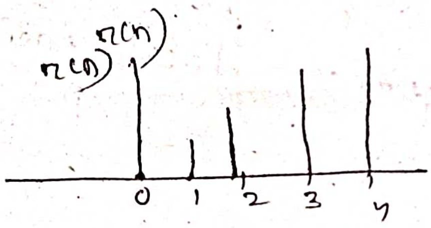
$$u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$



Unit Ramp Sequence:

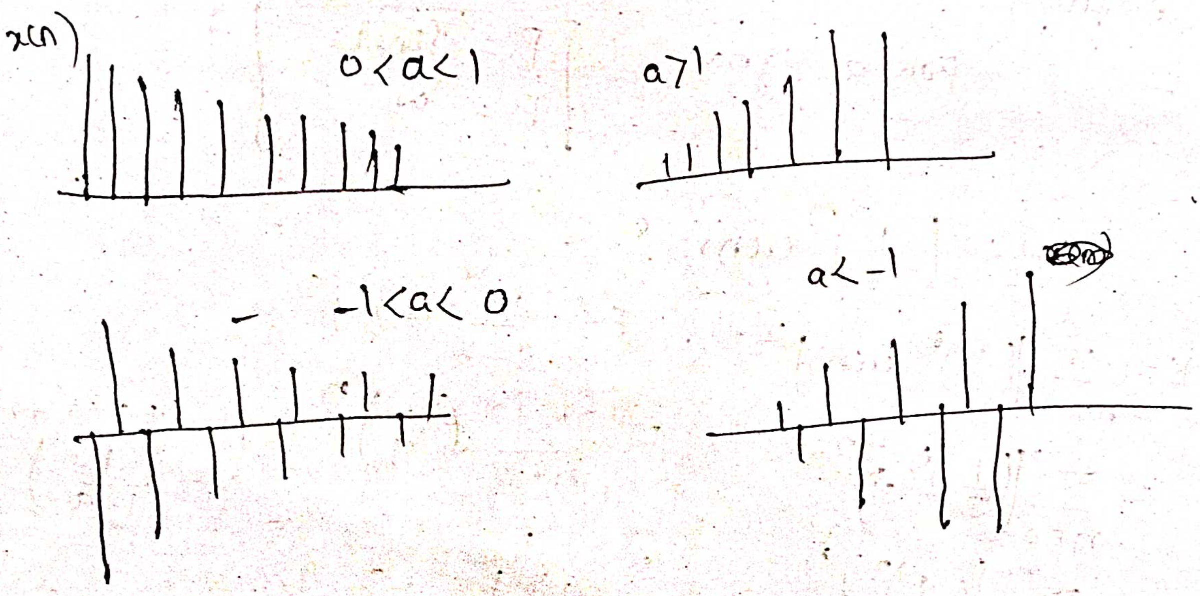
- It is denoted as $r(n)$.
- " " defined as

$$r(n) = n \quad n \geq 0$$
$$= 0 \quad n < 0$$



4. Exponential Sequence:

$$x(n) = a^n \quad \text{for all } n.$$



$\infty = \infty$

Classification of Discrete time Signal :

1. Energy & power signal

The energy E of a DTS signal x(n) is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The average power of DTS ~~x(n)~~ is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

- A signal is an energy signal if & only if the total energy of the signal is finite & average power = 0 .

Energy signal = $\boxed{\begin{array}{c}\text{total energy} = \text{finite} \\ P = 0\end{array}}$

- A signal is said to be power signal if signal is finite & average power of the energy is ∞ .

- Power signal . $\boxed{\begin{array}{l}P = \text{finite} \\ E = \infty\end{array}}$

Ex

$$x(n) = \left(\frac{1}{3}\right)^n u(n) .$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{3}\right)^n u(n)\right|^2$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{3}\right)^n \cdot 1\right]^2$$

$$= \sum_{n=0}^{+} \left[\left(\frac{1}{3}\right)^2\right]^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n \quad \left(1+a+a^2+a^3 = \frac{1}{1-a}\right)$$

$$= \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$$

②

The average power $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$ ②

$$\Rightarrow P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^n$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-\left(\frac{1}{9}\right)^{N+1}}{1-\frac{1}{9}} \right]$$

$$= 0$$

As E = finite
P = 0 } it is a energy signal.

2) $x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} \left| e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} \right|^2 = \infty \qquad \left(\because \left|e^{j(\omega+\theta)}\right| = 1\right)$$

$$= \sum_{n=-\infty}^{\infty} 1 = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} \right|^2 \qquad \left[\because \sum_{n=-N}^{N} 1 = 2N+1\right]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \lim_{N\to\infty} \frac{1}{(2N+1)} + (2N+1)$$

$$= 1$$

E = ∞
P = finite } Power signal.

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3) $x(n) = e^{2n} u(n)$

$$E = \sum_{n=-\infty}^{\infty} \left(e^{2n} u(n)\right)^2 = \sum_{n=0}^{\infty} \left|e^{2n}\right|^2 = \sum_{n=0}^{\infty} e^{4n} = 1 + e^4 + e^8 + \dots + \infty = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} e^{4n}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{e^{4(N+1)}-1}{e^4-1} \right]$$

Periodic & Aperiodic Signals

A signal is periodic with period N if & only if

$$x(n+N) = x(n) \quad \text{for all } n \quad -(1)$$

The smallest value of N for which equⁿ (1) holds is called fundamental period.

→ If equⁿ (1) does not satisfy even for one value of n then the discrete time signal is aperiodic.

A discrete time sinusoidal signal is given by

$$x(n) = A \sin(w_0 n + \theta) \quad \text{———} (2)$$

The unit of ω & θ are radians.

The signal x(n) is periodic if & only if $x(n) = x(n+N)$ for all n.

from equⁿ (2)

$$x(n+N) = A \sin[w_0(n+N) + \theta]$$

$$= A \sin[w_0 n + w_0 N + \theta) \quad -(3)$$

~~Compare~~ equⁿ (2) & (3) are equal if

$$w_0 N = 2\pi m$$

$$\Rightarrow w_0 = 2\pi\left(\frac{m}{N}\right)$$

Therefore discrete time signal is periodic if fundamental freq w_0 is rational multiple of 2π otherwise the discrete time signal is aperiodic.

→ Sum of two periodic signal is periodic if $\frac{N_1}{N_2}$ be a rational number otherwise sum is not periodic.

Ex

① $x(n) = e^{j6\pi n}$

$\omega_0 = 6\pi$.

Fundamental freq is multiple of π. Therefore signal is periodic.

$$\omega_0 = 2\pi\left(\frac{m}{N}\right)$$

$$\Rightarrow N = 2\pi\left(\frac{m}{\omega_0}\right)$$

$$\Rightarrow N = 2\pi\left(\frac{m}{6\pi}\right)$$

$$\Rightarrow N = \frac{m}{3}$$

if $m=3$ then $N=1$

The fundamental period = 1.

② $x(n) = e^{j\frac{3}{5}\left(n+\frac{1}{2}\right)}$

As ω_0 is not multiple of 2π i.e $\omega = \frac{3}{5}$ so it is aperiod.

③ $x(n) = \cos\left(\frac{2\pi}{3}\right)n$.

$\omega_0 = \frac{2\pi}{3}$.

signal is periodic,

fundamental period = $N = 2\pi\left(\frac{m}{\omega_0}\right)$

$$\Rightarrow N = 2\pi \frac{m}{2\pi/3}$$

$$\Rightarrow N = 3$$

(4) $x(n) = \cos\left(\frac{\pi}{3}\right)n + \cos\left(\frac{3\pi}{4}\right)n$

$\omega_{01} = \frac{\pi}{3}$ $\Rightarrow N_1 = 2\pi\left(\frac{m_1}{\pi/3}\right) \Rightarrow 6$ $(\because m_1=1)$

$\omega_{02} = \frac{3\pi}{4}$ $\Rightarrow N_2 = 2\pi\left(\frac{m_2}{3\pi/4}\right) \neq \frac{8}{3} m_2$

$= 8$

$\frac{N_1}{N_2} = \frac{6}{8} = \frac{3}{4}$ $\Rightarrow N = \frac{N_1}{N_2} = \frac{3}{4} \Rightarrow N = 3N_2 = 4N_1$

$\Rightarrow N = 24$

Causal & Noncausal Signal:

A signal is said to be causal if its value zero for $n<0$. Otherwise the signal is noncausal.

Ex: $x_1(n) = a^n u(n)$

$x_2(n) = \{\underset{\uparrow}{1}, 2, -1, 0, 2\}$ } causal

Ex: Noncausal signal:

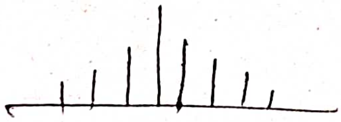
$x_1(n) = \{-1, 1, \underset{\uparrow}{4}, 2, 3, 2, 1\}$

$x_2(n) = a^n u(-n+1)$

Symmetric (Even) & Asymmetric (odd) signals:

A discrete (DTS) $x(n)$ is said to be symmetric (even) if it satisfy the condition

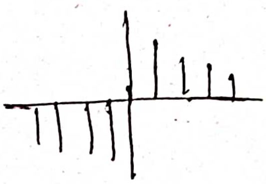
$\boxed{x(-n) = x(n)}$ for all n.



Ex: $x(n) = \cos wn$.

The signal is said to be an odd signal if it satisfy the condition.

$x(-n) = -x(n)$ for all n.



Ex: $x(n) = A \sin wn$.

if $x(n)$ is odd then $x(0) =$ _

A signal $x(n)$ can be expressed as sum of even & odd components. That is

$x(n) = x_e(n) + x_o(n)$ —(1)

Replace n by $-n$

$\Rightarrow x(-n) = x_e(-n) + x_o(-n)$

$\Rightarrow x(-n) = x_e(n) - x_o(n)$ —(2)

✓

Adding eq^n (1) & (2)

$$x(n) = x_e(n) + x_o(n)$$

$$+ \quad x(-n) = x_e(n) - x_o(n)$$

$$\Rightarrow 2x_e(n) = x(n) + x(-n)$$

$$\Rightarrow \boxed{x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]}$$

Similarly By subtracting

$$2x_o(n) = x(n) - x(-n)$$

$$\Rightarrow \boxed{x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]}$$

Manipulation of discrete time signal:

Signal processing is a group of basic operation applied to an input signal resulting in another signal as the output.

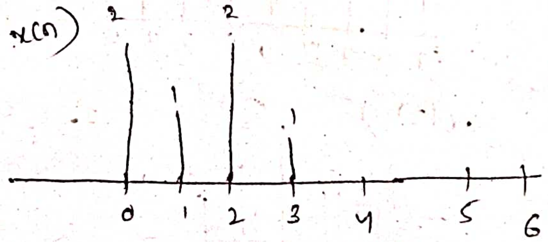
The basic operations are

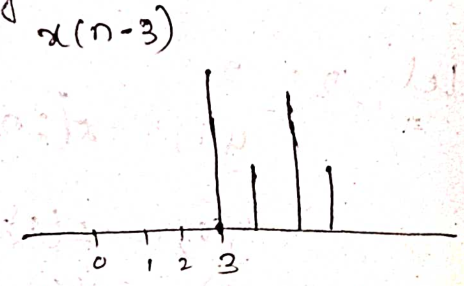
1. Shifting (2) Time reversal (3) Time scaling 4. Scalar multiplication 5. Signal multiplier 6. Signal addition.

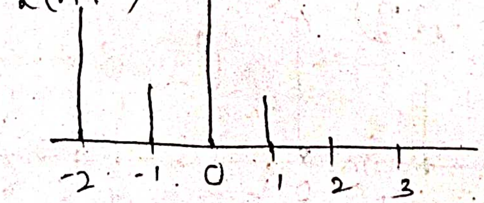
1. Shifting:

A signal $x(n)$ may be shifted in time replacing the independent variable n by $n-k$, where k is an integer, $y(n) = x(n-k)$

- If k is positive the shifting delays the sequence.
- If k is negative the shifting advances the sequence.





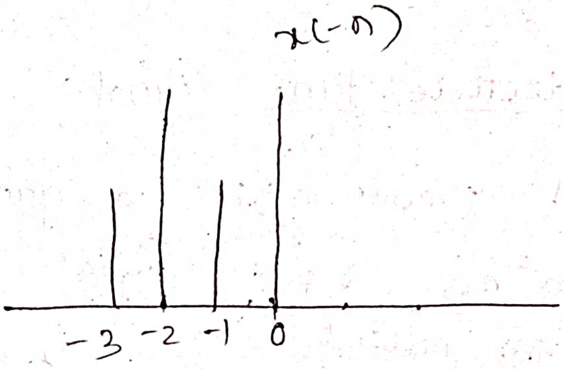


Time reversal:

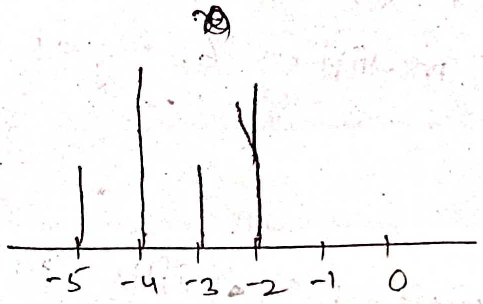
The time reversal can be obtained by ~~two units of time &~~ Folding the sequence about $n = 0$.

$X(n) - ..$

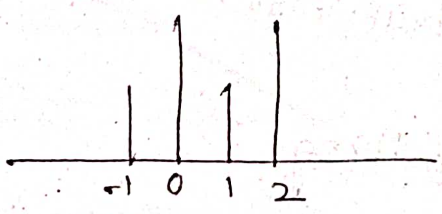
$x(-n) =$



$x(-n-2)$



$x(-n+2)$



Time scaling :

This is ~~accomplished~~ accomplished by replacing n by λn in the sequence $x(n)$.

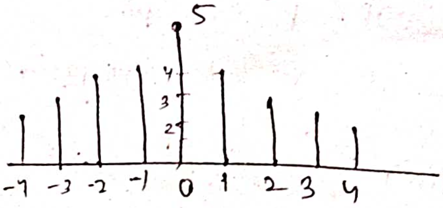
Let $\lambda = 2$.

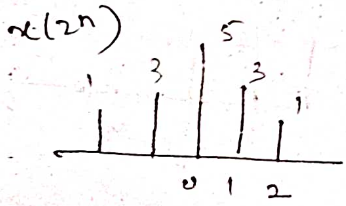
$y(n) = x(2n)$

$y(0) = x(0) = 5$

$y(1) = x(2) = 3$

$y(2) = x(4) = 1$



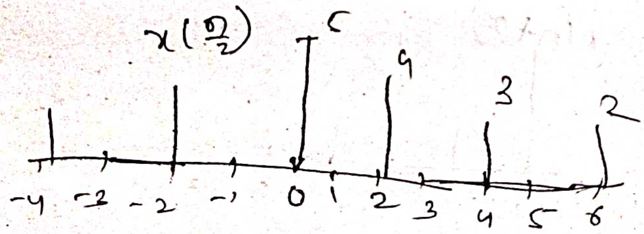


If $y(n) = x\left(\frac{n}{2}\right)$

$y(0) = x(0) = 5$

$y(1) = x\left(\frac{1}{2}\right) = 0$

$y(2) = x\left(\frac{2}{2}\right) = 4$



Scalar multiplication

Here the signal $x(n)$ is multiplied by a scale factor a.

$$x(n) \xrightarrow{a} y(n) = a \cdot x(n)$$

Ex: $x(n) = \{1, 2, 1, -1\}$.

$a = 2$

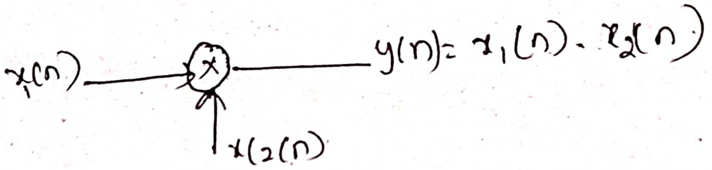
$\Rightarrow y(n) = a \cdot x(n)$

$= 2x(n)$

$\Rightarrow y(n) = \{2, 4, 2, -2\}$

Signal Multiplier:

The multiplication of two signal sequences to form another sequence.



let $x_1(n) = \{1, 2, 1, 1\}$

$x_2(n) = \{2, 1, 2, 1\}$

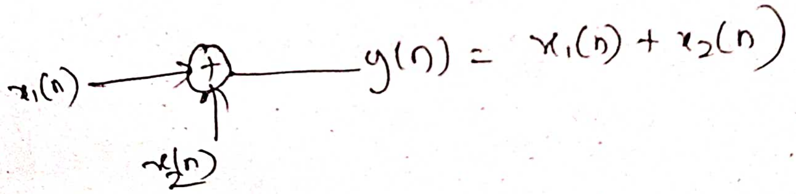
$y(n) = \{2, 2, 2, 1\}$

$y(n) = x_1(n) \cdot x_2(n)$

$= \{1 \cdot 2, 2 \cdot 1, 1 \cdot 2, 1 \cdot 1\}$

Addition Operation:

Two signals can added by using an adder.



$x_1(n) = \{1, 2, 1, 1\}$

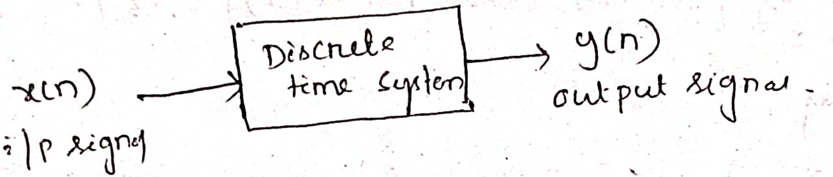
$x_2(n) = \{2, 1, 2, 1\}$.

$y(n) = x_1(n) + x_2(n) = \{1+2, 2+1, 1+2, 1+1\}$

$= \{3, 3, 3, 2\}$

Discrete - Time System:

A device or algorithm that operates on a discrete time input signal x(n), according to some well defined rule is known as discrete - time system.



$$\boxed{y(n) : T[x(n)]}$$

Here $x(n)$ = input signal
$y(n)$: Output Signal
$T \rightarrow$ Transformation.

Input - Output Description of system:

The input - Output description of a System consist of a mathematical expression on a rule, which explicitly defines the relation between the input & output signals.

$$x(n) \xrightarrow{T} y(n)$$

Ex. Let $x(n) = |n| \quad -3 \leq n \leq 3$
$\qquad 0 \quad$ otherwise.

i) Identity system $\Rightarrow y(n) = x(n)$
$y(n) = x(n) = \{-3, -2, -1, \underset{\uparrow}{0}, 1, 2, 3\}$

ii) unit delay system, $y(n) = x(n-1)$
$y(n) = \{-3, -2, \underset{\uparrow}{-1}, 0, 1, 2, 3\}$

iii) Unit advanu System ; $y(n) = x(n+1)$

$y(n) = \{-3, -2, -1, 0, \underset{\uparrow}{1}, 2, 3\}$

iv) moving average filter

$$y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$$

$$y(0) = \frac{1}{3}[x(-1) + x(0) + x(1)] = \frac{1}{3}[1+0+1] = \frac{2}{3}$$

Similarly

$$y(n) = [\ldots 0, 1, \frac{5}{3}, 2, 1, \frac{2}{3}, 1, 2, \frac{5}{3}, 1, 0]$$

v) Median filter - $y(n) = \text{median}[x(n-1), x(n, x(n+1)]$

$y(0) = x(-1), x(0), x(1)$

$\overset{\text{medn}}{=} \{2, 1, 0\} = 1$

vi) Accumulator = $y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \ldots$

By simple algebraic manipulation the input-output relation of the accumulator can be written as

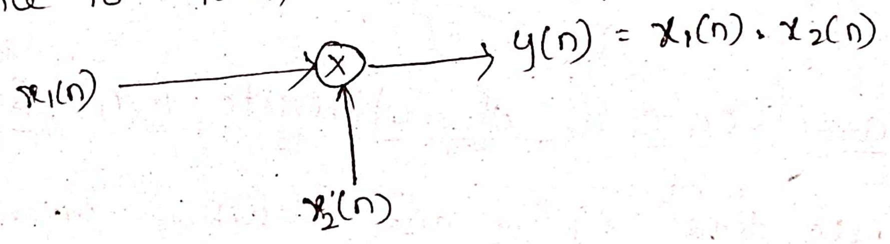
$$\boxed{y(n) = y(n-1) + x(n).}$$

Block diagram Representation of Discrete time Systems (DTS).

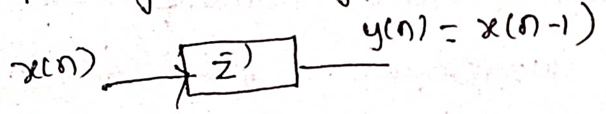
i) **Adder** : It performs the addition of two signal sequences to form another (the sum) sequence which denotes as $y(n)$.

ii) **A constant multiplier:** It represents applying a scale factor on the input $x(n)$.

iii) **Signal multiplier:** Multiplication of two signal sequence to form another sequence.

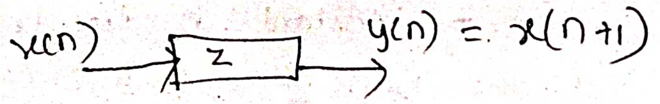


iv) **A unit delay element:** The unit delay is a special system that simply delays the signal passing through it by one sample.

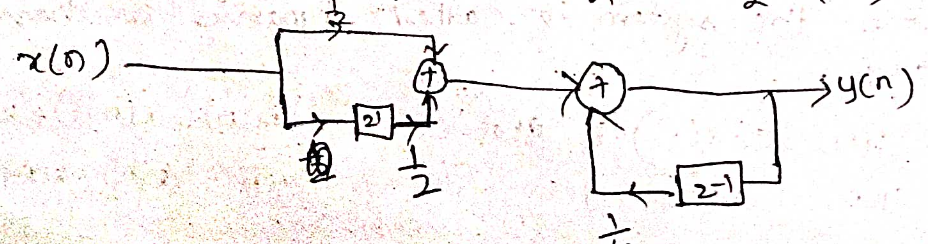


$$\boxed{y(n) = x(n-1)}$$

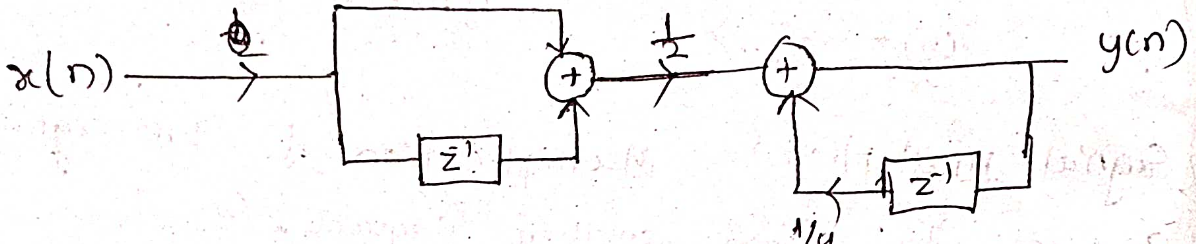
v) **A unit Advance element:** A unit advance moves the input $x(n)$ ahead by one sample in time to yield $x(n+1)$. The operator Z being used to denote the unit advance.



Ex: Design $y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$



ii) $y(n) = \frac{1}{4} y(n-1) + \frac{1}{2}[x(n) + x(n-1)]$



Classification of discrete time system :

Discrete time (DTS) systems are classified according to their general properties & characteristics. They are

i) Static & Dynamic System
ii) Time variant & time invariant
iii) Causal & Noncausal "
iv) Linear & Non linear "
v) FIR & IIR System.
vi) Stable & Unstable "

i) Static & Dynamic System :

→ A DTS is called static or memoryless if its output at any instant n depends on the input sample at the same time ; but not on past or future samples of the input.

→ Otherwise the system is called dynamic or have memory.

Ex: $y(n) = a \cdot x(n)$ $\}$ Static $\quad y(n) = x(n-1) + x(n-2)$

$y(n) = a x^2(n)$ $\quad y(n) = x(n) + x(n+1)$ $\}$ dynamic

11) Time variant & Time invariant :

A system is called time-invariant if its input-output c/s don't change with time.

Ex Let i/p sequence $x(n)$ o/p sequence $y(n)$ Now delay input sequence by k samples & O/p sequence becomes

$$y(n,k) = T[x(n-k)]$$

Delay o/p sequence by k samples, denotes it as $y(n-k)$

if $\boxed{y(n, k) = y(n-k)}$

for all possible value of k, the system is time-invariant.

Otherwise if the output $y(n,k) \neq y(n-k)$ even for one value of k, the system is time vaiant.

Ex ① $y(n) = x(n) + x(n-1)$

Given that $y(n) = T[x(n)]$
$= x(n) + x(n-1)$.

if input is delay by k sample.

$$y(n,k) = T[x(n-k)] = x(n-k) + x(n-k-1)$$

if we delay output by k sample.

$$y(n-k) = x(n-k) + x(n-k-1)$$

Hence $y(n,k) = y(n-k)$

so system is time invariant

② $y(n) = x(-n)$

$\Rightarrow y(n) = T[x(n)] = x(-n)$

input delay by k sample.

$y(n,k) = T[x(n-k)] = x(-n-k)$

output delay by k sample.

$y(n-k) = x[-(n-k)] = x(-n+k)$

As $y(n-k) \neq y(n,k)$, it is a time variant system.

③ $y(n) = n\,x(n)$

$\Rightarrow y(n) = T[x(n)] = n\,x(n)$

i/p delay by k samples

$y(n,k) = T[x(n-k)] = n\,x(n-k)$

o/p delay by k sample.

$y(n-k) = (n-k).x(n-k)$

$= n.\,x(n-k) - k.x(n-k)$

It is time variant.

~~③ Linear & Non linear System:~~

③ Causal & Non Causal System:

A system is said to be causal if the output of the system at any time n depends only on present & past inputs but does not depend on future inputs.

This can be represented mathematically as

$y(n) = F[x(n), x(n-1), x(n-2) \ldots]$

If a system depends not only on present & past inputs but also on future inputs then it is said to be a non causal system.

Ex ① $y(n) = x(n) + \frac{1}{x(n-1)}$

for $n=-1$, $y(-1) = x(n) + \frac{1}{x(-1-1)} = x(-1) + \frac{1}{x(-2)}$

$n=0$, $y(0) = x(0) + \frac{1}{x(-1)}$

$n=1$, $y(1) = x(1) + \frac{1}{x(0)}$

For all value of n, o/p depends on present & past i/p. So it is causal.

(11) $y(n) = x(n^2)$

$n=-1$, $y(-1) = x(1)$

$n=0$, $y(0) = x(0)$

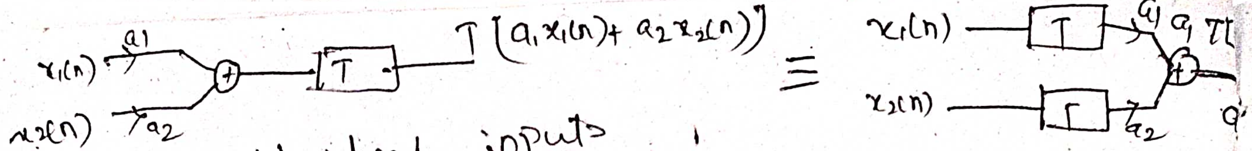
$n=1$, $y(1) = x(1)$

$n=2$, $y(2) = x(4)$

the system depends on future inputs. system is non - causal.

④ Linear & Non linear System:

- A system that satisfy the superposition principle is said to be a linear system.
- Superposition principle states that the response of the system to a weighted sum of signals to be equal to the corresponding weighted sum of the outputs of the system of each



the individual inputs

A system is Linear if & only if

$$\boxed{T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[}$$

for any arbitary constants a_1 & a_2

- A relaxed system that does not satis the superposition principle is called non-li

① $y(n) = x(n) + \frac{1}{x(n+1)}$

For two input sequence $x_1(n)$ & $x_2(n)$ the correspon o/p are

$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{x_1(n+1)}$

$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{x_2(n+1)}$

The output due to weighted sum input is

$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)]$

$= a_1 x_1(n) + a_2 x_2(n) + \frac{1}{a_1 x_1(n+1) + a_2 x_2(n+1)}$

The linear combination of two o/p sequence.

$a_1 y_1(n) + a_2 y_2(n)] = a_1 x_1(n) + \frac{a_1}{x_1(n+1)} + a_2 x_2(n) + \frac{a_2}{x_2(n+1)}$

It is non linear system.

② $y(n) = x^2(n)$

$y_1(n) = T[x_1(n)] = x_1^2(n)$

$y_2(n) = T[x_2(n)] = x_2^2(n)$

o/p due to weighted sum of input

~~weighted sum of o/p~~ .

$$T[a_1 x_1(n) + a_2 x_2(n)] = [a_1 x_1(n) + a_2 x_2(n)]^2$$

The o/p due to weighted sum of i/p.

$$y_3(n) = T[a_1 x_1(n)] + T[a_2 x_2(n)]$$

$$= a_1 x_1^2(n) + a_2 x_2^2(n)$$

System non linear -

③ $y(n) = n\,x(n)$.

Here $y_1(n) = n\,x_1(n)$

$y_2(n) = n\,x_2(n)$

$$T[a_1 x_1(n) + a_2 x_2(n)] = n[a_1 x_1(n) + a_2 x_2(n)]$$

$$= n a_1 x_1(n) + n a_2 x_2(n)$$

$$T[a_1 x_1(n)] + T[a_2 x_2(n)] = a_1 n x_1(n) + a_2 n x_2(n)$$

Linear system .

FIR & IIR System:

- If the impulse response sequence is of finite duration, the system is called as finite impulse Response (FIR) system.

Ex:- $h(n) = \begin{cases} -1 & n = 1, 2 \\ 1 & n = 4 \\ 0 & \text{otherwise.} \end{cases}$

- An infinite impulse response (IIR) system has an impulse response that is of infinite duration.

$$h(n) = a^n u(n)$$

stable and unstable system:

An LTI system is stable if it produces a bounded output sequence for every bounded input sequence.

- If for some bounded input sequence $x(n)$, the output is unbounded (infinite), the system is classified as unstable.

The necessary & sufficient condition for stability is

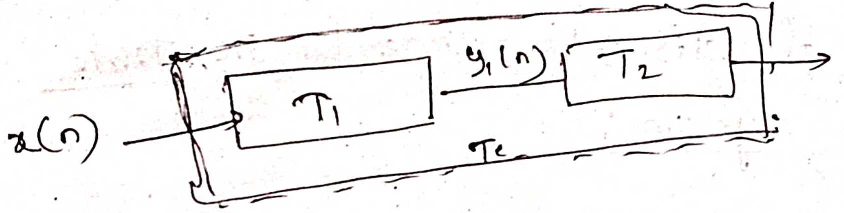
$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

Interconnection of Discrete Time System:

- Discrete time system can be interconnected to form larger system.

- There are two basic ~~types~~ ways in which system can be interconnected.
  - i) In cascade (series)
  - ii) In parallel.

Cascade Interconnection:



Here $y_1(n) = T_1[x(n)]$

$$y(n) = T_2\{T_1[(x(n))]\}$$
$$= T_2[y_1(n)]$$

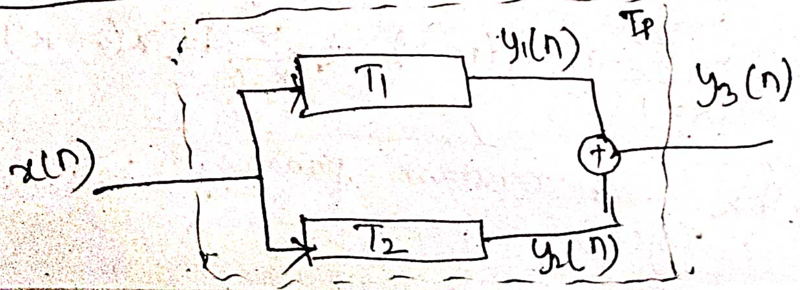
The system T_1 & T_2 can be combined or consolidated into a single overall system.

$$T_c \equiv T_1 \cdot T_2$$

The output can be written as

$$\boxed{y(n) = T_c[x(n)]}$$

ii) Parallel Interconnection:



Here $Y_3(n) = y_1(n) + y_2(n)$

$$= T_1[x(n)] + T_2[x(n)]$$

$$= (T_1 + T_2)[x(n)]$$

$$= T_P[x(n)]$$

Where $T_P = T_1 + T_2$

Analysis of discrete - Time Linear Time invariant systems:

Techniques for the Analyses of Linear Systems

✓ There are two basic methods for analyzing the behaviour or response of a linear system to a given i/p signal.

✓ One method is based on the direct solution of the i/p - o/p eqn for the system . which in general, has the form

$$y(n) = F[y(n-1), y(n-2) \ldots y(n-N), x(n), x(n), x(n-2) \ldots x(n-m))$$

when $F[\cdot]$ denotes some function of the quantities in blackets .

✓ For LTI system, general form of i/o relationship is

$$\boxed{y(n) = -\sum_{K=1}^{N} a_K\, y(n-K) + \sum_{K=0}^{M} b_K\, x(n-K)} \quad ①$$

when a_K, b_K are constant parameter & independent of $x(n)$ & $y(n)$

— The input-output relationship is called difference equ^n.

- The second method for analyzing the behavior of a linear system to a given input signal is first to decompose or resolve the input signal ~~to a~~ into a sum of elementary signals.

- Let the input signal $x(n)$ is resolved into a weighted sum of elementary signal components ~~[illegible]~~ $\{x_k(n)\}$ so that

$$x(n) = \sum_{k} c_k x_k(n)$$

where $\{c_k\}$ are the set of amplitudes (weighting coefficients) in the decomposition of the signal $x(n)$.

The total response to the input $x(n)$ is

$$y_k(n) = T[x_k(n)]$$

$$\boxed{y(n) = \sum_{k} c_k y_k(n)}$$

Resolution of discrete Time Signal into impulse:

An arbitary sequence $x(n)$ can be represented in terms of delayed & scaled impulse sequence $\delta(n)$.

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)}$$

Ex Represent $x(n) = \{4, 2, -1, 1, 3, 2, 1, 5\}$ as sum of shifted unit impulse.

$$x(n) = \{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$$

$n = -3, -2, -1, 0, 1, 2, 3, 4.$

$$x(n) = x(-3)\,\delta(n+3) + x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(0) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + x(3)\,\delta(n-3) + x(4)\,\delta(n-4)$$

$$= 4\,\delta(n+3) + 2\,\delta(n+2) - \delta(n+1) + \delta(n)$$

Saikam

①

Determine the impulse response of linear time invariant recursive System.

The general form of difference eqⁿ is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad —(1)$$

The solution of difference eqⁿ is

$$\boxed{y(n) = y_h(n) + y_p(n)}$$

where $y_h(n) \to$ homogeneous solution or natural response.
$y_p(n) \to$ Particular solution. or forced response.

Homogeneous solution is obtained by putting $x(n)$ to zero.

The general solution $y_h(n) = C_1 \lambda_1^n + C_2 \lambda_2^n + \dots C_N \lambda_N^n$

for roots like $\lambda_1 = 2$ & $\lambda_2 = 3$

$$y_h(n) = C_1 (2)^n + C_2 (3)^n$$

if λ_1 is repeated m times

$$\lambda_1^n (C_1 + C_2 n + C_3 n^2 + C_4 n^3 + \dots C_m n^{m-1})$$

for complex root $\lambda_1, \lambda_2 = a \pm jb$

$$y_h(n) = r^n (A_1 \cos n\theta + A_2 \sin n\theta)$$

$r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1} \frac{b}{a}$

- for impulse response, particular solution $y_p(n) = 0$

General form of particular solution for several types input

$x(n)$ input signal	$y_p(n)$ Particular solution
A (step input)	K
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} + \dots K_M$
$A^n n^M$	$A^n (K_0 n^M + K_1 n^{M-1} + \dots K_M)$
$A \cos \omega_0 n$ $A \sin \omega_0 n$	$K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$

① Determine the solution of difference equ^n

$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n)$ for $x(n) = 2^n u(n)$

Ans: The solution is $\boxed{y(n) = y_h(n) + y_p(n)}$

For input $x(n) = 2^n u(n)$ the particular solution is form of $\boxed{y_p(n) = K 2^n u(n)}$

So by substituting this in difference equ^n

$$K 2^n u(n) = \frac{5}{6} K 2^{n-1} u(n-1) - \frac{1}{6} K 2^{n-2} u(n-2) + 2^n u(n)$$

for $n = 2$

$$K 4 = \frac{5}{6} K . 2 - \frac{1}{6} K + 4$$

$$\Rightarrow 4K = \frac{10}{6} K - \frac{1}{6} K + 4$$

$$\Rightarrow 4K = \frac{9K}{6} + 4$$

$$\Rightarrow 4K = \frac{3K}{2} + 4 \Rightarrow 8K = 3K + 8 \Rightarrow 5K = 8 \Rightarrow K = \frac{8}{5}$$

Therefore $\boxed{y_p(n) = \frac{8}{5} 2^n u(n)}$

Homogeneous equ^n obtained by putting $y_h(n) = \lambda^n$ & $x(n) = 0$

$$\boxed{y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} . y(n-2) + x(n)} \quad \text{— ①}$$

$$\Rightarrow y(n) - \frac{5}{6} y(n-1) + \frac{1}{6} y(n-2) = 0$$

$$\Rightarrow \lambda^n - \frac{5}{6} \lambda^{n-1} + \frac{1}{6} \lambda^{n-2} = 0$$

$$\Rightarrow \lambda^{n-2}\left(\lambda^2 - \frac{5}{6} \lambda + \frac{1}{6}\right) = 0$$

⇒ $r^2 - \frac{5}{6}r + \frac{1}{6} = 0$

⇒ By solving this . $r_1 = \frac{1}{2}$, $r_2 = \frac{P}{3}$

$y_h(n) = c_1 (r_1)^n + c_2 (r_2)^n$

$= C\left(\frac{1}{2}\right)^n + c_2 \left(\frac{1}{3}\right)^n$

$y(n) = y_h(n) + y_p(n)$

⇒ $y(n) = c_1 \left(\frac{1}{2}\right)^n + c_2 \left(\frac{1}{3}\right)^n + \frac{8}{5} 2^n u(n)$

let $n = 0$, $y(0) = c_1 + c_2 + \frac{8}{5}$

$n = 1$ $y(1) = \frac{1}{2} c_1 + \frac{1}{3} c_2 + \frac{16}{5}$

In difference eqn substituting $y(-1)$ & $y(-2) = 0$ in eqn (1),

$y(0) = \frac{5}{6} y(0-1) - 1/6\, y(0-2) + x(0)$.

So $y(0) = \frac{5}{6} \cdot 0 - \frac{1}{6} 0 + x(0)$

⇒ $y(0) = x(0) = 1$

$y(1) = \frac{5}{6} \cdot y(0) + \frac{1}{6} y(-1) + x(1)$

⇒ $y(1) = \frac{5}{6} - 0 + 2$

⇒ $y(1) = \frac{5+12}{6} = \frac{17}{6}$ //

By substituting

$1 = c_1 + c_2 + \frac{8}{5}$

$\frac{17}{6} = \frac{1}{2} c_1 + \frac{1}{3} c_2 + \frac{16}{5}$

By solving $c_1 = -1$ & $c_2 = \frac{2}{5}$

$y(n) = \left(-\frac{1}{2}\right)^n u(n) + \frac{2}{5} \left(\frac{1}{3}\right)^n u(n) + \frac{8}{5} 2^n u(n)$

② Determine the solution of difference equⁿ

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) \quad — \quad (1)$$

where $x(n) = 2^n$ for $n \geq 0$

Ans The solution is $y(n) = y_h(n) + y_p(n)$

The particular solution $y_p(n) = k.2^n$

By substituting this

$$k.2^n - 3k2^{n-1} - 4k2^{n-2} = 2^n$$

for $n = 2$,

$$4k - 6k - 4k = 4$$

$$\Rightarrow k = -\frac{4}{6} = -\frac{2}{3}$$

so $y_p(n) = \left(-\frac{2}{3}\right).2^n$

Homogeneous solution: Substitute $y_h(n) = \lambda^n$, $x(n) = 0$

By substituting this

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

$$\Rightarrow \lambda^{n-2}(\lambda^2 - 3\lambda - 4) = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$\Rightarrow (\lambda + 4)(\lambda + 1) = 0$$

$\Rightarrow \lambda = +4, -1$ so $\lambda_1 = 4$, $\lambda_2 = -1$

$$\boxed{y_h(n) = c_1 4^n + c_2(-1)^n}$$

$y(0) = c_1 + c_2$

$y(1) = c_1 4 + c_2(-1)$

for $n = 0$ eqⁿ(1) becomes $y(0) - 3y(-1) - 4y(-2) = x(0)$

$\Rightarrow y(0) = x(0)\ 2\,0$

$n=1$,, $y(1) - 3y(0) - 4y(-1) = x(1)$

$\Rightarrow y(1) - 3.0 - 4y(-1) = 2^1$

$\Rightarrow y(1) = 2$

By substituting this

$0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

$2 = 4c_1 + (-1)c_2 \Rightarrow 2 = 4c_1 + (-1)(-c_1)$

$\Rightarrow 2 = 4c_1 + c_1$

$\Rightarrow 2 = 5c_1$

$\Rightarrow c_1 = \frac{2}{5}$

then $c_2 = -\frac{2}{5}$

So $y_h(n) = \left(\frac{2}{5}\right) \cdot 4^n + \left(-\frac{2}{5}\right)(-1)^n$

$y(n) = \left(\frac{2}{5}\right)4^n + \left(-\frac{2}{5}\right)(-1)^n + \left(-\frac{2}{3}\right) 2.^n$

Impulse Response and Convolution Sum :

A discrete - time system performs an operation on an input signal based on a predefined criteria to produce a modified output signal .

- The input signal $x(n)$ is the system excitation & $y(n)$ is the system response as shown in the fig .

- If the input to the system is a unit impulse i.e, $x(n) = \delta(n)$ then the output of the system is known as impulse response denoted by $h(n)$.

$$\boxed{h(n) = T[\delta(n)]}$$

For a linear time invariant system if the input sequence $x(n)$ & impulse response $h(n)$ then output $y(n)$ is given by equ

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)}$$

The above equ is known as convolution sum . It can be represented as

$$\boxed{y(n) = x(n) * h(n)}$$

where * denotes convolution ~~sum~~ operation .

The convolution sum of two sequence can be found by using following sequence .

Steps

1. Choose initial value of n, the starting time for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ & $h(n)$ starts at $n = n_2$ then $n = n_1 + n_2$ is a good choice .

2. Express both sequence in terms of index k .

3. Fold $h(k)$ about $k = 0$ to obtain $h(-k)$ & shift by n to the right if n is positive & left if n is negative to obtain $h(n-k)$.

4. Multiply the two sequences $x(k)$ & $h(n-k)$ element by element and sum the product to get $y(n)$.

5. Increment the index n, shift the sequence $h(n-k)$ to right by one sample & do step 4.

6. Repeat step 5 until sum of products is zero for all remaining values of n.

Properties of convolution:

1. Commutative Law: $x(n) * h(n) = h(n) * x(n)$

2. Associative Law: $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

3. Distributive Law: $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

Problem:

Determine the convolution sum of two sequences

$x(n) = \{3, 2, 1, 2\}$ $\quad h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

Ans: $x(n)$ starts at $n_1 = 0$ & $h(n)$ starts at $n_2 = -1$

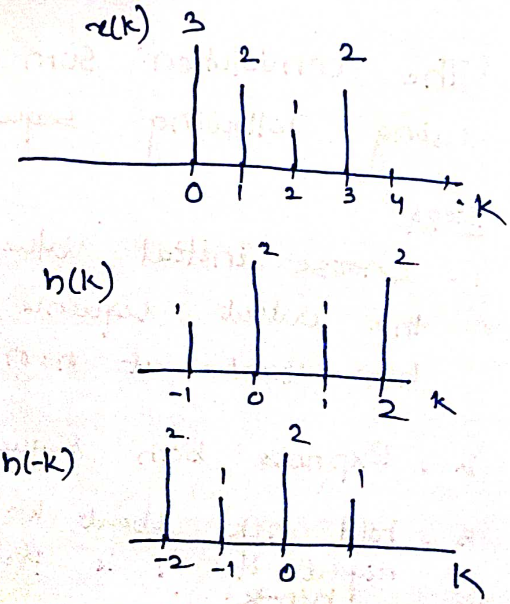
Therefore, the starting value of $n = n_1 + n_2 = -1$

for $n = -1$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k)$$

from fig we get

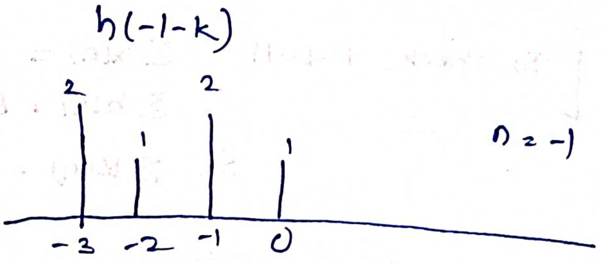
$y(-1) = 3.1 = 3$

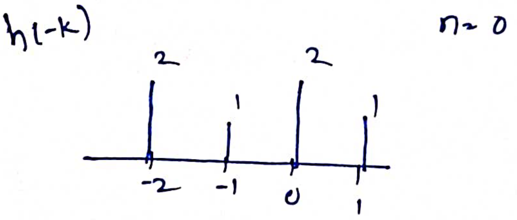


for $n = 0$

$$y(0) = \sum_{k=-\infty}^{\infty} x(h)h(-k)$$

$$= 3.2 + 2.1 = 8$$



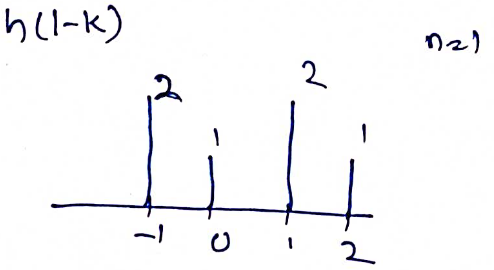


for $n = 1$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\,h(1-k)$$

$$= 3.1 + 2.2 + 1.1$$

$$= 8$$

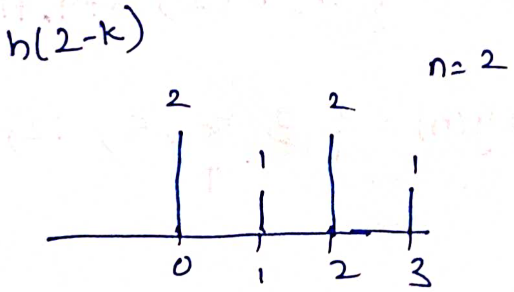


for $n = 2$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\,h(2-k)$$

$$= 3.2 + 2.1 + 1.2 + 2.1$$

$$= 12$$

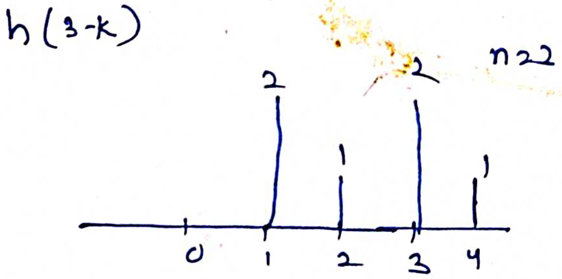


for $n = 3$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k)$$

$$= 3.0 + 2.2 + 1.1 + 2.2 + 0.1$$

$$= 9$$

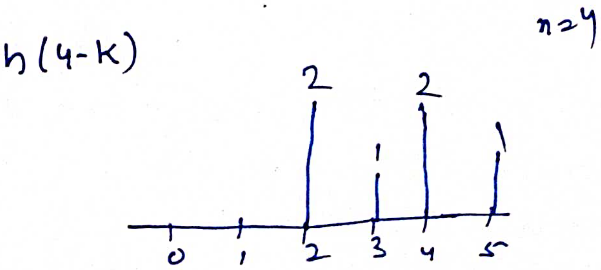


for $n = 4$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k)$$

$$= 1.2 + 2.1$$

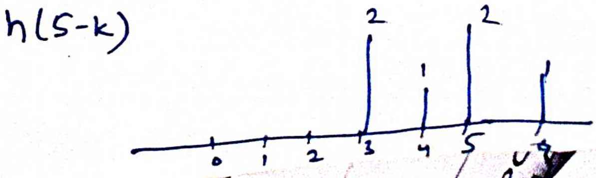
$$= 4$$



for $n = 5$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k)\,h(5-k)$$

$$= 2.2 = 4$$



$$y(n) = \{3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4\}$$

[To check Result $\Sigma x(n) = 8$ and $\Sigma y(n) =$ [illegible] $= 48$.

$\Sigma h(n) = 6$

So $\Sigma x(n) \cdot \Sigma h(n) = \Sigma y(n)$ Proved]

Method-2

x(n)

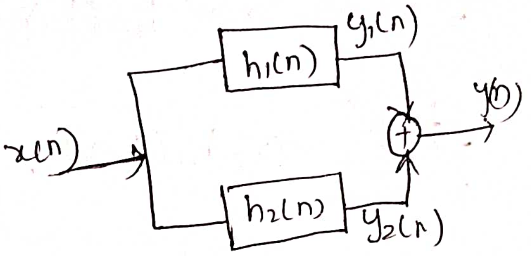
h(n)	3	2	1	2
1	3	2	1	2
2	6	4	2	4
1	3	2	1	2
2	6	4	2	4

$y(n) = \{ 3, 6+2, 3+4+1, 6+2+2+2, 4+1+4, 2+2, 4 \}$

$\Rightarrow y(n) = \{ 3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4 \}$

Interconnection of LTI system:

i) Parallal connection:



$$y(n) = y_1(n) + y_2(n)$$

$$y_1(n) = x(n) * h_1(n)$$
$$= \sum_{k=-\infty}^{\infty} x(k)\, h_1(n-k)$$

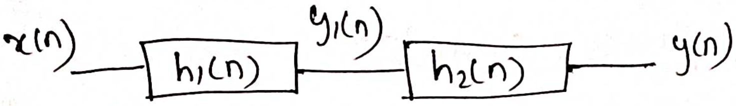
$$y_2(n) = x(n) * h_2(n)$$
$$= \sum_{k=-\infty}^{\infty} x(k)\, h_2(n-k)$$

$$y(n) = y_1(n) + y_2(n)$$
$$= \sum_{k=-\infty}^{\infty} x(k)\, h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k)\, h_2(n-k)$$
$$= \sum_{k=-\infty}^{\infty} x(k) \left[h_1(n-k) + h_2(n-k) \right]$$
$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$
$$= x(n) * h(n)$$

where $h(n-k) = h_1(n-k) + h_2(n-k)$

i.e $\boxed{h(k) \doteq h_1(n) + h_2(n)}$

ii) Cascade connection of two system:



$$y_1(k) = x(k) * h_1(k)$$
$$= \sum_{v=-\infty}^{\infty} x(v)\, h_1(k-v)$$

$$y(n) = y_1(k) * h_2(k)$$

$$= \left[\sum_{v=-\infty}^{\infty} x(v)\, h_1(k-v)\right] * h_2(k)$$

$$= \sum_{k=-\infty}^{+\infty}\sum_{v=-\infty}^{\infty} x(v)\, h_1(k-v) \,.\, \sum_{k=-\infty}^{\infty} h_2(n-k)$$

Let $k-v = p$

$$\Rightarrow y(n) = \sum_{v=-\infty}^{\infty} x(v) \sum_{p=-\infty}^{\infty} h_1(p)\, h_2(n-v-p).$$

$$= \sum_{v=-\infty}^{\infty} x(v)\, h(n-v)$$

$$= x(n) * h(n)$$

Where $h(n) = \sum_{k=-\infty}^{\infty} h_1(k)\, h_2(n-k)$

$$= h_1(n) * h_2(n)$$

Correlation of two Sequence:

- correlation is basically used for compare two signals.
- Correlation is a measure of degree to which two signals are similar.
- Correlation of two signals is divided into

  i) Cross Correlation

  ii) Auto correlation.

i) Cross correlation:

The cross - correlation betn a pair of signal x(n) & y(n) is given by

$$\boxed{\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l)}$$

The index (l) is the shift (lag) parameter.

Here x(n) is the reference sequence remain unshifted in time whereas y(n) is shifted l units in time w.r.t x(n).

Also we can write

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-1)$$

$$= \sum_{n=-\infty}^{\infty} y(n+l) \cdot x(n)$$

if $l = 0$, $r_{xy}(0) = r_{yx}(0) = \sum_{n=-\infty}^{\infty} x(n)\, y(n)$.

Also $\boxed{r_{xy}(l) = x(l) * y(-l)} = \sum_{n=-\infty}^{\infty} x(n)\, y[-(l-n)]$

Auto Correlation:

The autocorrelation of a sequence is correlation of a sequence with itself. The autocorrelation of a sequence x(n) is defined by

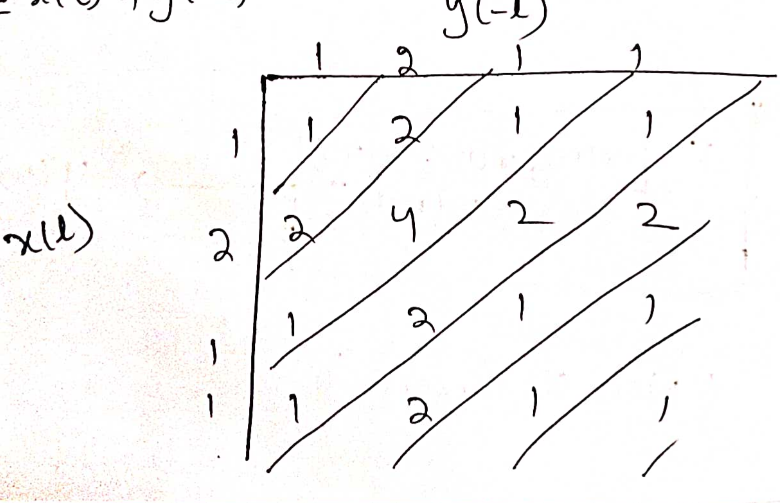
$$\boxed{r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l)} \Rightarrow r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\, x(n)$$

if $l = 0$, $\boxed{r_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2(n)}$

Q. Find cross correlation ~~of~~ two finite sequence
$x(n) = \{1,2,1,1\}$, $y(n) = \{1,1,2,1\}$

Ans $x(n) = \{1,2,1,1\}$, $y(l) = \{1,1,2,1\}$, $y(-l) = \{\underset{\bullet}{1},2,1,\underset{\uparrow}{1}\}$

$r_{xy}(l) = x(l) * y(-l)$



	y(-l)			
	1	2	1	1
x(l) 1	1	2	1	1
2	2	4	2	2
1	1	2	1	1
1	1	2	1	1

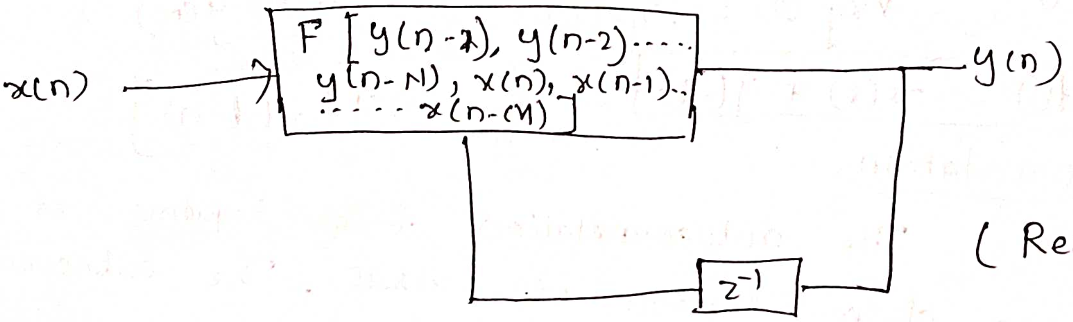
$r_{xy}(l) = \{1,4,6,6,5,2,1\}$

Discrete Time System describe by difference eqⁿ :

Reccursive & Non Recursive discrete time System :

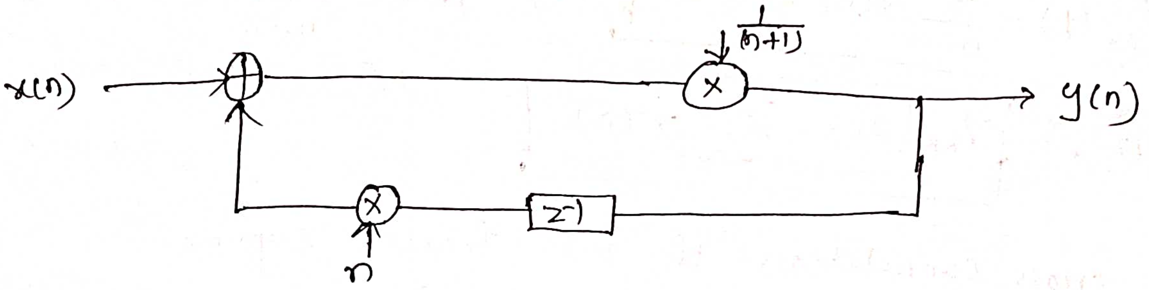
- There are many systems where it is either necessary or desirable to express the output of the system not only in terms of present & past value of input but also available past output values.

- A System whose output y(n) at time n depends on any number of past output values y(n-1), y(n-2) .... is called a recursive system.

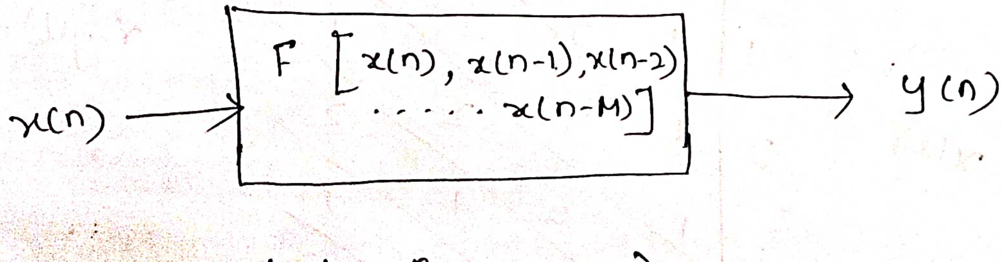


(Recursive)

Ex: $y(n) = \frac{n}{n+1} y(n-1) + \frac{1}{n+1} x(n)$



- A System whose output y(n) depends only on the present & past input is known as non-recursive.



(Non Recursive)

Z transform

Z transform :

The Z transform is a powerful mathematical tool for analysis of linear time invariant discrete time systems in the frequency domain.

The Z transform of a discrete time signal x(n) is defined as

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}} \qquad \text{—— (1)}$$

$$\Rightarrow X(z) = Z\{x(n)\}$$

where z is a complex variable. In polar form z can be expressed as $z = re^{j\omega}$ when r = radious of circle.

$$\boxed{X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, r^{-n} e^{-j\omega n}}$$

ROC: The region of convergence (ROC) of X(z) is the set of all values of z for which X(z) attains a finite value.

Properties of ROC:

1. The ROC is a ring on disk in z plane centered at origin.
2. The ROC cannot contain any pole.
3. If x(n) is causal sequence, ROC is the entire z place except z=0,
4. If x(n) is non " " , " " " " " " z=∞
5. If x(n) is a finite two sided sequence the ROC is the entire z plane except z = 0 & z = ∞.
6. The ROC must be a connected region.
7. The ROC of a LTI stable system contains the unit circle.
8. If x(n) is an infinite duration, two sided sequence the ROC will consider of a ring in the z plane, bounded on the interior & exterior by a pole, not containing any pole.

Examples: Find z transform & Roc .

① $x(n) = \{\underset{\uparrow}{1}, 0, 3, -1, 2\}$

$X(z) = \sum_{n=0}^{4} x(n)\, z^{-n}$

$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4}$

$= 1 + 0 + 3z^{-2} - 1 z^{-3} + 2z^{-4}$

$= 1 + 3z^{-2} - z^{3} + 2z^{-4}$

Here X(z) finite for all value of z except z = 0

② $x(n) = \{-3, -2, -1, 0, \underset{\uparrow}{1}\}$

$X(z) = \sum_{-4}^{0} x(n)\, z^{-n}$

$= x(0) z^{0} + x(-1) z^{1} + x(-2) z^{2} + x(-3) z^{3} + x(-4) z^{4}$

$= 1 + 0.z^{1} - 1 z^{2} - 2z^{3} - 3z^{4}$

$= 1 - z^{2} - 2z^{3} - 3z^{4}$

Here X(z) convergences all value of z except z = ∞ ,

③ $x(n) = \{2, -1, 3, 2, \underset{\uparrow}{1}, 0, 3, -1, 1\}$

$X(z) = \sum_{n=-4}^{4} x(n)\, z^{-n}$

$= x(-4) z^{4} + x(-3) z^{3} + x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1}$
$+ x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4}$

$= 2z^{4} - z^{3} + 3z^{2} + 2z^{1} + 1.z^{0} + 0 + 3z^{-2} - 1z^{-3} + 1z^{-4}$

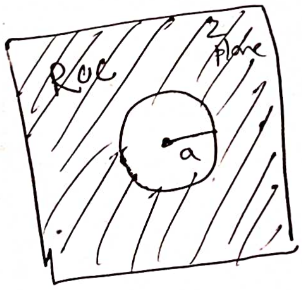
$= 2z^{4} - z^{3} + 3z^{2} + 2z + 1 + 3z^{-2} - z^{-3} + z^{-4}$

Here X(z) converges all value of x except z = 0 &
z = ∞ .

transform & Roc of infinite duration Sequence.

Determine z transform & Roc of $x(n) = a^n u(n)$

$x(n) = a^n u(n)$



$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \frac{1}{1-az^{-1}} \qquad \left(\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\right)$$

$$= \frac{1}{1-\frac{a}{z}} = \frac{1}{\frac{z-a}{z}} = \frac{z}{z-a}$$

Here $|az^{-1}| < 1$

$\Rightarrow \left|\frac{a}{z}\right| < \cancel{1} \Rightarrow \boxed{a < |z|} \Rightarrow |z| > a$

X(z) converge all value of z ~~except~~ i.e Roe is $|z| > a$

② $x(n) = -b^n u(-n-1)$. Find z transform & its Roc.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Here $u(-n-1) = 1$ for $n \nleq 1$
$= 0$ otherwise

$$= -\sum_{n=-\infty}^{-1} b^n . z^{-n}$$

$$= -\sum_{n=1}^{\infty} b^{-n} z^{n} = -\left[\sum_{n=0}^{\infty} b^{-n} z^{n} - 1\right]$$

$$= -\sum_{n=0}^{\infty} (b^{-1}z)^n - 1 = \frac{1}{1-b^{-1}z} - 1 \qquad \text{when } |b^{-1}z| < 1$$

$$X(Z) = -\left[\frac{1}{1-b^{-1}Z} - 1\right]$$

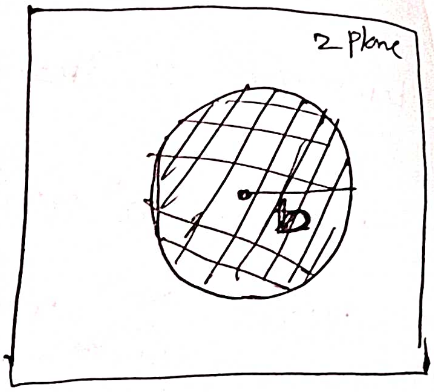
$$= -\left[\frac{1}{1-\frac{Z}{b}} - 1\right]$$

$$= -\left[\frac{1}{\frac{b-Z}{b}} - 1\right]$$

$$= -\left[\frac{b}{b-Z} - 1\right]$$

$$= -\left[\frac{b-b+Z}{b-Z}\right]$$

$$= -\left[\frac{Z}{b-Z}\right] = \frac{Z}{Z-b}$$



Roc : $|b^{-1}z| < 1$

$\Rightarrow \left|\frac{Z}{b}\right| < 1$

$\Rightarrow |Z| < b$

The Roc is the interior of a circle having radius b.

③ Find z transform & ROC of the signal

$x(n) = a^n u(n) + b^n u(-n-1)$

Ans Z transform is given by

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) . z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[a^n u(n) + b^n u(-n-1)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=1}^{\infty} b^{-n} z^n$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n + \left[\sum_{n=0}^{\infty} b^{-n} z^n - 1\right]$$

Plane

$$= \frac{1}{1-\left|\frac{a}{z}\right|} + \left[\frac{1}{1-\frac{z}{b}} - 1\right]$$

$$= \frac{1}{\frac{z-a}{z}} + \left[\frac{1}{\frac{b-z}{b}} - 1\right]$$

$$= \frac{z}{z-a} + \left[\frac{b}{b-z} - 1\right]$$

$$= \frac{z}{z-a} + \frac{b-b+z}{b-z}$$

$$= \frac{z}{z-a} + \frac{z}{b-z}$$

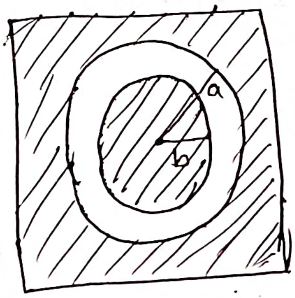
$$X(z) = \frac{z}{z-a} - \frac{z}{z-b}$$

$\left|\frac{a}{z}\right| < 1$ $\left|\frac{z}{b}\right| < 1$

$\Rightarrow a < |z|$ $\Rightarrow |z| < b$

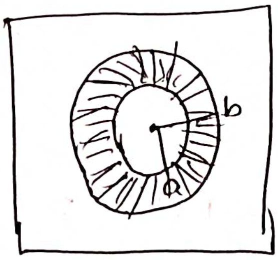
$\Rightarrow |z| > a$

$a < |z| < b$



$b < a$

$b > a$



Properties of z transform:

i) **Linearity** :

If $Z\{x_1(n)\} \rightarrow X_1(z)$

$Z\{x_2(n)\} = X_2(z)$

Then $Z\{a x_1(n) + b x_2(n)\} = a X_1(z) + b X_2(z)$

Proof $Z\{a x_1(n) + b x_2(n)\} = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$

$$= \sum_{n=-\infty}^{\infty} a x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} b x_2(n) z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= a X_1(z) + b X_2(z)$$ (Proved)

ii) Time shifting :

If $Z\{x(n)\} = X(z)$

the $Z\{x(n-m)\} = z^{-m} X(z)$

Proof !

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-m) \,.\, z^{-n} \,.\, z^{-m} \,.\, z^{m}$$

$$= z^{-m} \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n} . z^{m}$$

$$= z^{-m} \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-(n-m)}$$

Let $p = n-m$.

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p)\, z^{-p}$$

$$= z^{-m} X(z)$$

(Proved.)

iii) Time Scaling :

If $Z\{x(n)\} = X(z)$

then $Z\{a^{n} x(n)\} = X(a^{-1}z)$

Proof

$$Z\{a^{n} x(n)\} = \sum_{n=-\infty}^{\infty} a^{n} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(a^{-1} z\right)^{-n}$$

$$= X(a^{-1} z)$$

(Proved)

) **Time Revensal :**

$$\text{If} \quad z\{x(n)\} = X(z)$$

$$\text{Then} \quad z\{x(-n)\} = X(z^{-1})$$

Proof :

$$z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(p)\, z^{p} \qquad \text{Let } p=-n$$

$$= \sum_{p=-\infty}^{\infty} x(p)(z^{-1})^{-p}$$

$$= X(z^{-1})$$

v) **Differentiation:**

$$\text{If} \quad z\{x(n)\} = X(z)$$

$$\text{then} \quad z\{n\,x(n)\} = -z\,\frac{d\,X(z)}{dz}$$

Proof : We know that $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

differentiate the z transform

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-n)\, z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-n)\, z^{-n} \cdot z^{-1}$$

$$\Rightarrow \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n)\,(-n)\, z^{-n}\,\frac{1}{z}$$

$$= z\,\frac{d}{dz} X(z) = (-n) \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= -z\,\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n\,x(n)\, z^{-n} = z\{n\,x(n)\}$$ (Proof

(VI) Convolution Theorem:

If $Z\{x(n)\} = X(z)$

$Z\{h(n)\} = H(z)$

then $Z\{x(n) * h(n)\} = X(z) \cdot H(z)$.

Proof:

Let $y(n) = x(n) * h(n)$

$$= \sum_{K=-\infty}^{\infty} x(k)\, h(n-k)$$

$$Z\{x(n) * h(n)\} = \sum_{n=-\infty}^{\infty} \left[\sum_{K=-\infty} x(k)\, h(n-k) \right] z^{-n}$$

$$= \sum_{K=-\infty}^{\infty} x(K) \sum_{n=-\infty}^{\infty} h(n-k)\, z^{-n} \cdot z^{-K} \cdot z^{K}$$

$$= \sum_{K=-\infty}^{\infty} x(K)\, z^{-K} \cdot \sum_{n=-\infty}^{\infty} h(n-k)\, z^{-(n-k)}$$

Let $n-K = p$, $\Rightarrow n = K+p$

$$= \sum_{K=-\infty}^{\infty} x(K)\, z^{-K} \sum_{p=-\infty}^{\infty} h(p)\, z^{-p}$$

$$= X(z) \cdot H(z)$$

Proved

(VII) Correlation of two Sequence:

If $Z\{x_1(n)\} = X_1(z)$

$Z\{x_2(n)\} = X_2(z)$

Then $Z[\gamma_{x_1 x_2}(l)] = X_1(z)\, X_2(z^{-1})$

Proof

$$Z[\gamma_{x_1 x_2}(l)] = Z\left[\sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2(n-l) \right]$$

$$= Z[x_1(l) * x_2(-l)]$$

$$= X_1(z) \cdot X_2(z^{-1})$$

(By Convolution Property) & Time Reversal.

~~M) Multiplication of Two sequences:~~

Rational Z transform:

i) Poles & Zeros:

→ The zeros of a z transform $X(z)$ are the values of z for which $X(z) = 0$.

→ The poles of a z transform are the values of z for which $X(z) = \infty$.

Let $X(z) = \frac{B(z)}{A(z)} = \frac{b_0}{a_0} z^{-M+N} \frac{(z-z_1)(z-z_2)\dots(z-z_m)}{(z-p_1)(z-p_2)\dots(z-p_N)}$

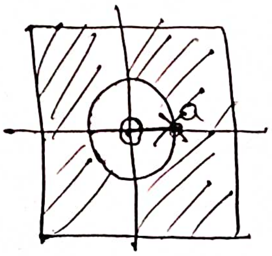
$$\Rightarrow \boxed{X(z) = G z^{N-M} \cdot \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}}$$

Where $G \equiv \frac{b_0}{a_0}$, $A(z)$ & $B(z)$ are polynomials in z.

Here $X(z)$ has M finite zeros at $z = z_1, z_2 \dots z_m$ &
N finite poles at $z = p_1, p_2 \dots p_N$.

Ex $x(n) = a^n u(n)$ find pole & zero.

So $X(z) = \frac{z}{z-a}$ Here $p_1 = a$
$z_1 = 0$



System Function of a Linear Time Invariant & Inverse

From the convolution Property

$$Y(z) = X(z) \cdot H(z)$$

$$\Rightarrow \boxed{H(z) = \frac{Y(z)}{X(z)}}$$

The transform H(z) is called system function.

We can write $H(z) = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$

~~If~~ → The zeros of the system function H(z) are the values of z for which H(z) = 0

→ The poles of the system function H(z) are the values of z for which H(z) = ∞ .

* If $a_k = 0$ then $\boxed{H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}}$

Here H(z) contains M zeros & Mth order pole is at z=0 . Such system is called 'all zero system'.

* If $b_k = 0$, then $\boxed{H(z) = \dfrac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}}}$

Here H(z) consists of N poles & Nth order zero at origin z=0 . Such system is called "all pole system".

Inverse Z transform:

There are four methods that often used for evaluation of inverse z transform.

1. Long division Method
2. Partial fraction expansion method
3. Residue or Contour Integration method
4. Convolution method.

1. By Partial Fraction Expansion Method:

1. Find inverse transform of

$$X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} \qquad |z| > 2.$$

Ans. First eliminate negative power by multiplying z^2 both numerator & denominator.

$$\Rightarrow X(z) = \frac{z^2+3z}{z^2+3z+2}$$

$$\Rightarrow X(z) = \frac{z(z+3)}{(z+1)(z+2)}$$

divide $X(z)$ by z.

$$\frac{X(z)}{z} = \frac{z+3}{(z+1)(z+2)} \quad \text{------} \quad (1)$$

equ^n (1) can be written in partial fraction.

$$\frac{X(z)}{z} = \frac{c_1}{(z+1)} + \frac{c_2}{(z+2)}$$

Evaluating c_1 & c_2.

$$c_1 = (z+1)\frac{X(z)}{z}\Big|_{z=-1}$$

$$= \cancel{(z+1)}\frac{(z+3)}{\cancel{(z+1)}(z+2)}\Big|_{z=-1} = \frac{z+3}{z+2}\Big|_{z=-1} = \frac{2}{1} = 2$$

$$C_2 = (z+2)\,\frac{X(z)}{z}\Big|_{z=-2}$$

$$= \cancel{(z+2)}\,\frac{(z+3)}{(z+1)\,\cancel{(z+2)}}\Big|_{z=-2}$$

$$= \frac{z+3}{z+1}\Big|_{z=-2} = \frac{1}{-1} = -1$$

Therefore $\frac{X(z)}{z} = \frac{2}{z+1} - \frac{1}{z+2}$

$$\Rightarrow \frac{X(z)}{\cancel{z}} = 2\,\frac{z}{z+1} - \frac{z}{z+2}$$

As ROC $|z| > 2$ sequence in causal

$$x(n) = 2(-1)^n u(n) - (-2)^n u(n)$$

② $X(z) = \frac{z(z^2-4z+5)}{(z-1)(z-2)(z-3)}$ Roc $2<|z|<3$

$$\Rightarrow \frac{X(z)}{z} = \frac{z^2-4z+5}{(z-1)(z-2)(z-3)}$$

$$= \frac{C1}{(z-1)} + \frac{C2}{(z-2)} + \frac{C3}{(z-3)}$$

$$C_1 = (z-1)\,\frac{X(z)}{z}\Big|_{z=1} = \cancel{(z-1)}\,\frac{(z^2-4z+5)}{\cancel{(z-1)}(z-2)(z-3)}\Big|_{z=1}$$

$$= \frac{z^2-4z+5}{\cancel{(z-1)}(z-2)(z-3)}\Big|_{z=1}$$

$$= \frac{1-4+5}{(-1)(-2)} = \frac{2}{2} = \underline{\underline{1}}$$

$$C_2 = (z-2)\,\frac{X(z)}{z}\Big|_{z=2} = \frac{\cancel{(z-2)}\,(z^2-4z+5)}{(z-1)\,\cancel{(z-2)}\,(z-3)}\Big|_{z=2} = \frac{z^2-4z+5}{(z-1)(z-3)}\Big|_{z=2}$$

$$= \frac{4-8+5}{(1)(-1)} = \frac{1}{-1} = \underline{\underline{-1}}$$

$$c_3 = (z-3)\frac{X(z)}{z}\Big|_{z=3} \Rightarrow c_3 = \cancel{(z-3)}\frac{(z^2-4z+5)}{(z-1)(z-2)\cancel{(z-3)}}\Big|_{z=3} = \frac{z^2-4z+5}{(z-1)(z-2)}\Big|_{z=3}$$

$$\Rightarrow c_3 = \frac{9-12+5}{(2)\cdot(1)} = \frac{2}{2} = 1$$

$$\frac{X(z)}{z} = \frac{1}{(z-1)} - \frac{1}{(z-2)} + \frac{1}{(z-3)}$$

$$\Rightarrow X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{z-3}$$

$$\Rightarrow x(n) = 1^n u(n) - 2^n u(n) - 3^n u(-n-1)$$

① If Roc is $|z| > 3$ then

$$x(n) = u(n) - (2)^n u(n) + 3^n u(n)$$

② If Roc $|z| < 1$ then

$$x(n) = -u(-n-1) + 2^n u(-n-1) - (3)^n u(-n-1)$$

Residue Method :

The inverse z transform relation is given by contour integral

$$\boxed{x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz}$$

Where c is a circle in the z plane in the Roc of $X(z)$

The above equ can be written as

$$x(n) = \Sigma\,[\text{residues of } X(z)\,z^{n-1} \text{ at the pole inside } c]$$

$$= \sum_i (z-z_i)\,X(z)\,z^{n-1}\Big|_{z=z_i}$$

① Using Residue method find inverse z transform of

$$X(z) = \frac{z+1}{(z+0.2)(z+1)}, \quad |z| > 1$$

Ans We know that

$$x(n) = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz$$

$$= \Sigma \text{ residue of } X(z).\,z^{n-1} \text{ at pole of } X(z).\,z^{n-1} \text{ within } c.$$

$$= \Sigma \text{residue of } \frac{(z+1)}{(z+0.2)(z+1)} \cdot z^{n-1} \text{ at pole of same within } c.$$

Here ROC is $|z| > 1$ & pole at $z = -0.2$ & $z = 1$.

for $n = 0$.

$$x(0) = \sum \text{residue of } \frac{(z+1)}{(z+0.2)(z-1)} z^{(0-1)} \quad \text{at } ~~\text{poles zero}~~ ,$$

$$= \sum \text{residue of } \frac{(z+1)}{(z+0.2)(z-1)} \cdot z^{-1}$$

$$= \sum \text{residue of } \frac{(z+1)}{z(z+0.2)(z-1)} \quad \text{at pole } z=0,\ z=-0.2\ \&\ z=1.$$

$$= \frac{z\cdot(z+1)}{\cancel{z}(z+0.2)\cdot(z-1)}\bigg|_{z=0} + \frac{\cancel{(z+0.2)}(z+1)}{z\cancel{(z+0.2)}(z-1)}\bigg|_{z=-0.2}$$

$$+ \frac{\cancel{(z-1)}(z+1)}{z(z+0.2)\cancel{(z-1)}}\bigg|_{z=1}$$

$$= \frac{1}{-0.2} + \left(\frac{0.8}{0.24}\right) + \frac{2}{1(1+0.2)} = \frac{1}{-0.2} + \frac{.8}{.24} + \frac{2}{1.2}$$

$$= -5 + \frac{10}{3} + \frac{5}{3} = \frac{-15+10+5}{3} = \frac{0}{3} = 0.$$

$x(0) = 0$.

for $n \geq 1$

$$x(n) = \sum \text{residue of } \frac{(z+1)}{(z+0.2)(z-1)} \cdot z^{n-1} \quad \text{at pole } z=-0.2,\ z=1$$

~~$= \frac{(z+1)\, z^{n} \cdot z^{-1}}{(z+0.2)(z-1)} = \frac{(z+1)\cdot z^{n}}{z(z+0.2)(z-1)}$ at pole for $z = 0, -0.2$ & 1~~

$$= \frac{\cancel{(z+0.2)}(z+1)}{\cancel{(z+0.2)}(z-1)} z^{n-1}\bigg|_{z=-0.2} + \frac{\cancel{(z-1)}(z+1)}{(z+0.2)\cancel{(z-1)}} \cdot z^{n-1}\bigg|_{z=1}$$

$$= \frac{z^{n-1}(z+1)}{(z-1)}\bigg|_{z=-0.2} + \frac{z^{n-1}(z+1)}{(z+0.2)}\bigg|_{z=1}$$

$$x(n) = \frac{(-0.2+1)}{(-0.2-1)} \cdot (-0.2)^{n-1} + \frac{(1+1)}{(1+0.2)} \, 1^{n-1}$$

$$= -\frac{.8}{1.2}(-0.2)^{n-1} + \frac{2}{1.2} \, 1^{n-1}$$

$$= -\frac{8}{12}(-0.2)^{n-1} + \frac{20}{12} \, 1^{n-1}$$

$$= -\frac{2}{3}(-0.2)^{n-1} + \frac{5}{3} \, 1$$

$$= -\frac{2}{3}(-0.2)^{n-1} + \frac{5}{3}$$

$$\Rightarrow x(n) = -\frac{2}{3}(-0.2)^{n-1} u(n-1) + \frac{5}{3} u(n-1)$$

② Find inverse z transform using residue method.

$$X(z) = \frac{z}{(z-2)(z-3)} \quad |z| < 2.$$

Ans. Here two poles $z = 3$ & $z = 2$ outside the ROC $|z| < 2$

So it is non causal.

for $n < 0$

$x(n) = -\sum$ residue of $X(z) \cdot z^{n-1}$ at pole $z = 2$ & $z = 3$

$$= -\left[\cancel{(z-2)} \frac{z \cdot z^{n-1}}{\cancel{(z-2)}(z-3)} \bigg|_{z=2} + \cancel{(z-3)} \frac{z \cdot z^{n-1}}{(z-2)\cancel{(z-3)}} \bigg|_{z=3} \right]$$

~~$= -\left[\frac{z \cdot z^{n-1}}{(2-3)} + \frac{z \, z^{n-1}}{(3-2)} \right] = -\left[-z z^{n-1} + z \cdot z^{n-1} \right]$~~

$$= -\left[\frac{z^n}{z-3} \bigg|_{z=2} + \frac{z^n}{z-2} \bigg|_{z=3} \right] \qquad \left(\because z \cdot z^{n-1} = z^1 \cdot z^n \cdot z^{-1} = z^n \right)$$

$$= -\left[-(2^n) + 3^n \right]$$

$$x(n) = 2^n - 3^n$$

$x(n)$ can be written as

$$x(n) = (2^n - 3^n) \, u(-n-1) \qquad \text{as } n < 0$$

Using Residue method Find inverse Ztransform of

$$X(Z) = \frac{Z}{(Z-1)(Z-2)} \qquad 1<|Z|<2$$

Ans ~~Here the pole Z=1 lies inside & Z=2~~

The contour integration C lies in the annular region of ROC,

$x(n)$ = Σ residue of $X(Z) \cdot Z^{n-1}$ at pole $Z = 1$ & $Z = 2$.

$$\Rightarrow x(n) = \cancel{(Z-1)}\frac{Z \cdot Z^{n-1}}{\cancel{(Z-1)}(Z-2)}\Bigg|_{Z=1} - \frac{\cancel{(Z-2)} \cdot Z \cdot Z^{n-1}}{(Z-1)\cancel{(Z-2)}}\Bigg|_{Z=2}$$

(∵ − sign for pole z = 2 due to outside of ROC)

$$\Rightarrow x(n) = \frac{Z^n}{Z-2}\Bigg|_{Z=1} - \frac{Z^n}{Z-1}\Bigg|_{Z=2}$$

$[\because Z \cdot Z^{n-1} = \cancel{Z} \cdot Z^n \cdot \cancel{Z^{-1}}]$

$$= \frac{(1)^n}{1-2} - \frac{2^n}{2-1} = -(1)^n - 2^n$$

∴ $x(n)$ can be written as.

$$x(n) = -u(n) - 2^n(-n-1)$$

Ans

① Determine the causal signal x(n) having z transform $X(Z) = \frac{1}{(1-2Z^{-1})(1-z^{-1})^2}$ by Partial Fraction.

Ans

$$X(Z) = \frac{1}{\left(1-\frac{2}{Z}\right)\left(1-\frac{1}{Z}\right)^2} = \frac{1}{\frac{(Z-2)}{Z}\left(\frac{Z-1}{Z}\right)^2} = \frac{1}{\frac{(Z-2)}{Z}\frac{(Z-1)^2}{Z^2}}$$

$$\Rightarrow X(Z) = \frac{Z^3}{(Z-2)(Z-1)^2}$$

$$\Rightarrow \frac{X(Z)}{Z} = \frac{Z^2}{(Z-2)(Z-1)^2}$$

$$\Rightarrow \frac{X(Z)}{Z} = \frac{C_1}{(Z-2)} + \frac{C_2}{(Z-1)} + \frac{C_3}{(Z-2)^2}$$

$$C_1 = \frac{X(Z)}{Z}\cdot(Z-2)\Big|_{Z=2}$$

$$= (Z-2)\frac{Z^2}{(Z-2)(Z-1)^2}\Big|_{Z=2} = 4$$

$$C_2 = \frac{d}{dZ}\left[\frac{X(Z)}{Z}\cdot(Z-1)^2\right]_{Z=1}$$

$$= \frac{d}{dZ}\left[\frac{(Z-1)^2Z^2}{(Z-2)(Z-1)^2}\right|_{Z=1} = \frac{(Z-2)2Z-Z^2}{(Z-2)^2}\Big|_{Z=1} = -3$$

$$C_3 = (Z-1)^2\frac{X(Z)}{Z}\Big|_{Z=1}$$

$$= \cancel{(Z-1)^2}\frac{Z^2}{(Z-2)\cancel{(Z-1)^2}}\Big|_{Z=1} = -1$$

$$\frac{X(Z)}{Z} = \frac{4}{4-2} - \frac{3}{Z-1} - \frac{1}{(Z-1)^2}$$

$$\Rightarrow X(Z) = \frac{4Z}{(Z-2)} - \frac{3Z}{Z-1} - \frac{Z}{(Z-1)^2}$$

$$x(n) = 4(2)^n u(n) - 3u(n) - nu(n)$$

$X(z) = \frac{z+1}{(z+0.2)(z-1)}$ for $|z| > 1$

$x(n) = \sum$ residue of $X(z)\,z^{n-1}$ at the pole inside c

$= \sum_i (z - z_i)\, X(z)\, z^{n-1} \Big|_{z=z_i}$

Here $z_1 = -0.2$, $z_2 = 1$

$x(n) = \cancel{(z+0.2)} \frac{z+1}{\cancel{(z+0.2)}(z-1)} \cdot z^{n-1} \Big|_{z=-0.2} + (z-1) \frac{(z+1)}{(z+0.2)(z-1)} z^{n-1} \Big|_{z=1}$

$= \frac{z+1}{z-1} \cdot z^{n-1} \Big|_{z=-0.2} + (z-1)$

$X(z) = \frac{1 - \frac{1}{4} z^{-1}}{1 - \frac{1}{9} z^{-2}}$ Roc $|z| > \frac{1}{3}$

$= \cancel{\frac{1 - \frac{4}{z}}{1 - \frac{1}{9z^2}}} = \cancel{\frac{\frac{z-4}{z}}{\frac{9z^2-1}{9z^2}}} = \cancel{\frac{z-4}{9z^2-1}}$

Multiply z^2 in both N & D

$= \frac{z^2 - \frac{1}{4} z}{z^2 - \frac{1}{9}} = \frac{z\left(z - \frac{1}{4}\right)}{z^2 - \frac{1}{9}} = \frac{z\left(z - \frac{1}{4}\right)}{\left(z + \frac{1}{3}\right)\left(z - \frac{1}{3}\right)}$

$z_i = -\frac{1}{3}, \frac{1}{3}$

$$\left.\frac{\cancel{\left(z+\frac{1}{3}\right)}\; z\left(z-\frac{1}{4}\right)}{\cancel{\left(z+\frac{1}{3}\right)}\left(z-\frac{1}{3}\right)}\cdot z^{n-1}\right|_{z=-\frac{1}{3}} + \left.\frac{\cancel{\left(z-\frac{1}{3}\right)}\; z\left(z-\frac{1}{4}\right)}{\left(z+\frac{1}{3}\right)\cancel{\left(z-\frac{1}{3}\right)}}\cdot z^{n-1}\right|_{z=\frac{1}{3}}$$

$$= \left.\frac{z\left(z-\frac{1}{4}\right)\cdot z^{n-1}}{\left(z-\frac{1}{3}\right)}\right|_{z=-\frac{1}{3}} + \left.\frac{z\left(z-\frac{1}{4}\right)}{\left(z+\frac{1}{3}\right)}\cdot z^{n-1}\right|_{z=\frac{1}{3}}$$

$$= \left.\frac{\left(z-\frac{1}{4}\right)}{\left(z-\frac{1}{3}\right)}\, z^{n}\right|_{z=-\frac{1}{3}} + \left.\frac{\left(z-\frac{1}{4}\right)}{\left(z+\frac{1}{3}\right)}\cdot z^{n}\right|_{z=\frac{1}{3}}$$

$$= \frac{-\frac{1}{3}-\frac{1}{4}}{-\frac{1}{3}-\frac{1}{3}}\left(-\frac{1}{3}\right)^{n} + \frac{\left(\frac{1}{3}-\frac{1}{4}\right)}{\left(\frac{1}{3}+\frac{1}{3}\right)}\left(\frac{1}{3}\right)^{n}$$

$$= \frac{\frac{-4-3}{12}}{-\frac{2}{3}}\left(-\frac{1}{3}\right)^{n} + \frac{\frac{4-3}{12}}{\frac{2}{3}}\left(\frac{1}{3}\right)^{n}$$

$$= \frac{\frac{-7}{12}}{\frac{-2}{3}}\left(-\frac{1}{3}\right)^{n} + \frac{\frac{1}{12}}{\frac{2}{3}}\left(\frac{1}{3}\right)^{n}$$

$$= +\frac{7}{12}\left(\frac{3}{2}\right)\left(-\frac{1}{3}\right)^{n} + \frac{1}{12}\times\frac{3}{2}\left(\frac{1}{3}\right)^{n}$$

$$= \frac{21}{24}\left(-\frac{1}{3}\right)^{n} + \frac{3}{24}\left(\frac{1}{3}\right)^{n}$$

$$= \frac{7}{8}\left(-\frac{1}{3}\right)^{n} + \frac{1}{8}\left(\frac{1}{3}\right)^{n}$$

$$x(n) = \frac{1}{8}\left(\frac{1}{3}\right)^{n} u(n) + \frac{7}{8}\left(-\frac{1}{3}\right)^{n} u(n).$$

Problems on System function:

1. Compute Poles, Zeros & System response of following

$$y(n) = 2y(n-1) + 3x(n).$$

Ans: $y(n) = 2y(n-1) + 3x(n)$

Taking z transform in both sizes

$$Y(z) = 2z^{-1}Y(z) + 3X(z)$$

$$\Rightarrow Y(z) - 2z^{-1}Y(z) = 3X(z)$$

$$\Rightarrow Y(z)[1-2z^{-1}] = 3X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{3}{1-2z^{-1}}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{3}{1-2z^{-1}}$$

Multiply z in both numerator & denominator

$$H(z) = \frac{3z}{z-2}$$

Here Pole, $P = 2$

Zero, $z = 0$

2. Find System function & impulse response of the System described by the differenu equation

$$y(n) = \frac{1}{5} y(n-1) + x(n)$$

Ans. Given $y(n) = \frac{1}{5} y(n-1) + x(n)$

Taking Z transform both sides

$$Y(z) = \frac{1}{5} z^{-1} Y(z) + X(z)$$

$$\Rightarrow Y(z) - \frac{1}{5} z^{-1} Y(z) = X(z)$$

$$\Rightarrow Y(z) \left(1 - \frac{1}{5} z^{-1}\right) = X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{5} z^{-1}}$$

$$\Rightarrow H(z) = \frac{1}{1 - \frac{1}{5} z^{-1}}$$

$$\Rightarrow H(z) = \frac{z}{z - \frac{1}{5}}$$ (Multiple z both numerator & denominator)

By using inverse z transfm

$$h(n) = \left(\frac{1}{5}\right)^n u(n)$$

x

Determine the Pole-zero plot for the system described by differenu equ^n $y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1)$

Ans: Given the difference equation

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1)$$

Taking z transform both the side

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = X(z) - z^{-1} X(z)$$

$$\Rightarrow Y(z) \left[1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2} \right] = X(z) \left[1 - z^{-1} \right]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

Multiply z^2 both numeratr & denominator

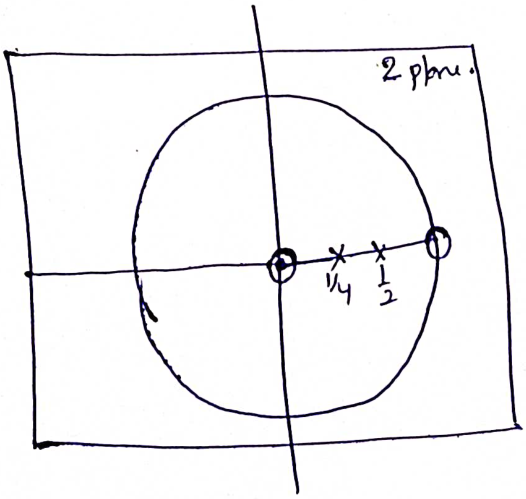
$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{z^2 - z}{z^2 - \frac{3}{4} z + \frac{1}{8}}$$

$$\Rightarrow \boxed{H(z) = \frac{z(z-1)}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}}$$

The Pole-Zero plot

Zeros: $z = 0, 1$

Poles: $P = \frac{1}{4}, \frac{1}{2}$



saiRam

Fourier Transform

Concept of discrete Fourier Transform:

→ Discrete Fourier Transform is a powerful computational tool for performing frequency analysis of discrete - time signal.

Discrete Fourier Transform (DFT):

* The DFT of a finite duration sequence x(n) is obtained by sampling the Fourier transform $X(e^{jw})$ at N equally spaced point over the interval $0 \leq w \leq 2\pi$ .with spacing $\frac{2\pi}{N}$.

→ Let $\underline{\underline{L}}$ is the Length of the sequence x(n) & DFT of the sequence x(n) obtained by sampling Fourier transform at $\underline{\underline{N}}$ $\underline{\text{equally}}$ ~~interval~~ spaced point.

- If N≥L we get a periodic sequence xp(n) over the interval 2π.
- If N<L, the replica of x(n) overlap & xp(n) is not identical to x(n).
- The DFT, denoted by X(k) is defined as

$$X(K) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \qquad 0 \leq k \leq N-1$$

$$x(n) = xp(n) \qquad 0 \leq n \leq N-1$$
$$= 0 \qquad \text{otherwise.}$$

So $$\boxed{DFT = X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}} \qquad 0 \leq K \leq N-1$$

$$\boxed{IDFT = x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{j2\pi kn/N}} \qquad 0 \leq n \leq N-1$$

$$X(K) = DFT[x(n)]$$

$$x(n) = IDFT[X(k)]$$

Example:

1. Find DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ & IDFT of $Y(K) = \{1, 0, 1, 0\}$

Ans: Let assume $L = N = 4$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nK/N} \qquad K = 0, 1, \ldots N-1$$

$$X(0) = \sum_{n=0}^{3} x(n)\, e^{-j2\pi n.0/4}$$

$$= \sum_{n=0}^{3} x(n) = X(0) + X(1) + X(2) + X(3)$$

$$= 1 + 0 + 0 + 0 = 2$$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j2\pi n.1/4} = \sum_{n=0}^{3} x(n)\, e^{-j\frac{\pi n}{2}}$$

$$= X(0) + X(1)\, e^{-j\pi/2} + X(2)\, e^{-j\pi} + X(3)\, e^{-j\pi\frac{3}{2}}$$

$$= \cancel{1 + \ldots + 0 + 0} . = 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + 0 + 0$$

$$= 1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n} = X(0) + X(1)\, e^{-j\pi} + X(2)\, e^{-j2\pi} + X(3)\, e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi$$

$$= 1 - 1 = 0$$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j3n\pi/2} = X(0) + X(1)\, e^{-j\frac{3\pi}{2}} + X(2)\, e^{-j3\pi} + X(3)\, e^{-j\frac{9\pi}{2}}$$

$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$= 1 + j$$

$$\boxed{X(K) = \{2, 1-j, 0, 1+j\}}$$

Given $Y(K) = \{1, 0, 1, 0\}$

$$y(n) = \frac{1}{N} \sum_{K=0}^{N-1} Y(K)\, e^{\frac{j2\pi nk}{N}} \qquad n = 0, 1, \dots N-1$$

$$y(0) = \frac{1}{4} \sum_{K=0}^{3} Y(K)\, e^{\frac{j2\pi 0.k}{4}}$$

$$= \frac{1}{4} \sum_{K=0}^{3} Y(K) = \frac{1}{4}\left[Y(0) + Y(1) + Y(2) + Y(3)\right]$$

$$= \frac{1}{4}\left[1 + 0 + 1 + 0\right] = \frac{1}{4} \times 2 = 0.5$$

$$y(1) = \frac{1}{4} \sum_{K=0}^{3} Y(K)\, e^{\frac{j2\pi K}{4}} = \frac{1}{4} \sum_{K=0}^{3} Y(K)\, e^{\frac{j\pi K}{2}}$$

$$= \frac{1}{4}\left[Y(0) + Y(1)\, e^{j\pi/2} + Y(2)\, e^{j\pi} + Y(3)\, e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos\pi + j\sin\pi + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 - 1 + 0\right] = 0$$

$$y(2) = \frac{1}{4} \sum_{K=0}^{3} Y(K)\, e^{\frac{j2\pi 2K}{4}} = \frac{1}{4} \sum_{K=0}^{3} Y(K)\, e^{j\pi K}$$

$$= \frac{1}{4} \cancel{\Sigma}\; Y(0) + Y(1)\, e^{j\pi} + Y(2)\, e^{j2\pi} + Y(3)\, e^{j3\pi}$$

$$= \frac{1}{4}\left[1 + 0 + \cos 2\pi + j\sin 2\pi + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 + 1 + 0\right] = 0.5$$

$$y(3) = \frac{1}{4} \sum_{K=0}^{3} Y(K)\, e^{\frac{j2\pi 3K}{4}} = \frac{1}{4} \sum_{K=0}^{3} Y(K)\, e^{\frac{j\pi 3K}{2}}$$

$$= \frac{1}{4}\left[Y(0) + Y(1)\, e^{\frac{j3\pi}{2}} + Y(2)\, e^{j3\pi} + Y(3)\, e^{j9\pi/2}\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos 3\pi + j\sin 3\pi + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 + (-1) + 0\right] = 0$$

$$\boxed{y(n) = \{0.5, 0, 0.5, 0\}}$$

Circular Convolution: i) Concentric circle Method:

Given two sequence $x_1(n)$ & $x_2(n)$, the circular convolution of these two sequence $x_3(n) = x_1(n) \textcircled{N} x_2(n)$ is found from following steps.

1. Graph N samples of $x_1(n)$ as equally spaced points around an outer circle in counterclockwise direction.
2. Start at the point as $x_1(n)$ graph N samples of $x_2(n)$ as equally spaced points around an inner circle ie clockwise direction.
3. Multiply corresponding samples on the two circles & sum the product to produce output.
4. Rotate the inner circle one sample at a time in counterclockwise direction & go to step 3 to obtain the next value of output.
5. Repeat step 4 until inner circle first sample lines up with the first sample of the exterior circle once again.

ii) Matrix Multiplication Method:

In this method circular convolution can be obtained by

$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x(N-1) & \cdots & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & \cdots & x_2(4) & x_2(3) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_2(N-2) & x_2(N-3) & x_2(N-4) & \cdots & x_2(0) & x_2(N-1) \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ x_1(N-2) \\ x_1(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ \vdots \\ x_3(N-2) \\ x_3(N-1) \end{bmatrix},$$

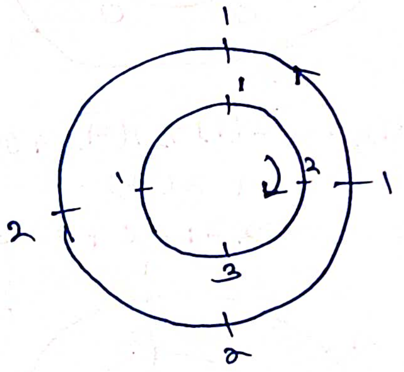
ii) Matrix Method:

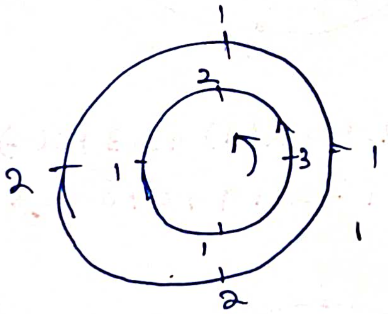
$$\begin{array}{ccc} x_2(n) & x_1(n) & y(n) \\ \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = & \begin{bmatrix} 1(1)+0(-1)+0(-2)+3.3+2.1 = 8 \\ 2(1)+1.(-1)+0.(-2)+0.(3)+3(-1) = -2 \\ 3(1)+2(-1)+1(-2)+0.3+0.(-1) = -1 \\ 0(1)+3.(-1)+2.(-2)+1.3+0.(-1) = -4 \\ 0(1)+0.(-1)+3.(-2)+2(3)+1(-1) = -1 \end{bmatrix} \end{array}$$

$y(n) = \{8, -2, -1, -4, -1\}$

② Find circular convolution of two sequences $x_1(n) = \{1, 2, 2, 1\}$ $x_2(n) = \{1, 2, 3, 1\}$

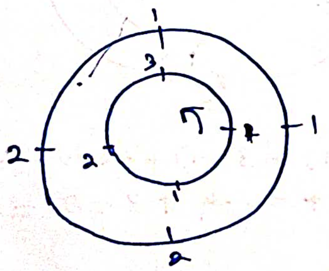
Ans

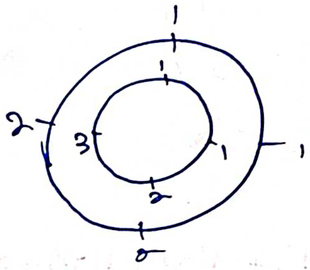




$y(1) = 2.1 + 1.2 + 1.2 + 3.1$
$= 9$

$y(0) = 1.1 + 1.2 + 3.2 + 2.1$
$= 11$





$y(3) = 1.1 + 3.2 + 2.2 + 1.1 = 12$

$y(2) = 3.1 + 2.2 + 1.2 + 2.1 = 10$

So $y(n) = \{11, 9, 10, 12\}$

Matrix Method

$$\begin{array}{ccc} x_2(n) & x_1(n) & y(n) \\ \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = & \begin{bmatrix} 11 \\ 9 \\ 10 \\ 12 \end{bmatrix} \end{array}$$

⑤

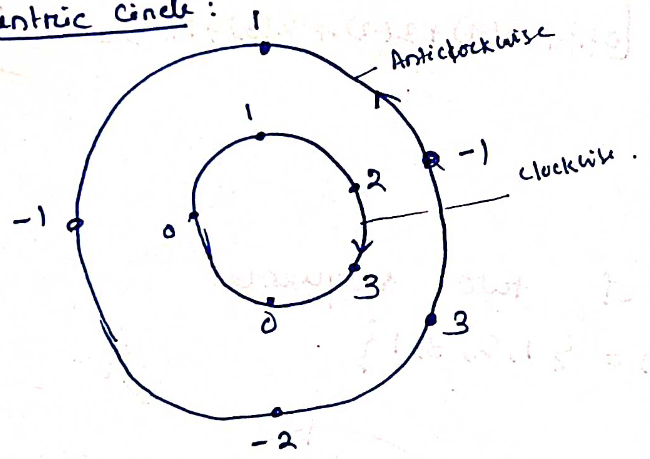
Q. Find the circular convolution of two finite sequence $x_1(n) = (1, -1, -2, 3, -1)$ $x_2(n) = \{1, 2, 3\}$

Ans For circular convolution, length of both sequence must be same.

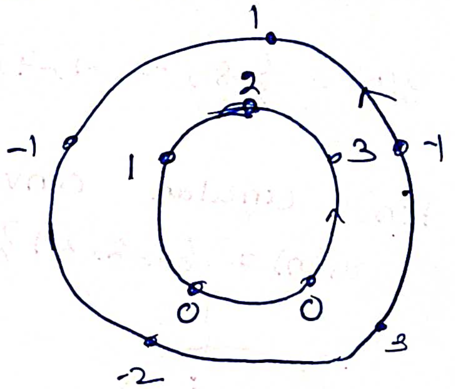
$x_1(n) = \{1, -1, -2, 3, -1\}$

$x_2(n) = \{1, 2, 3, 0, 0\}$

i) Concentric circle :

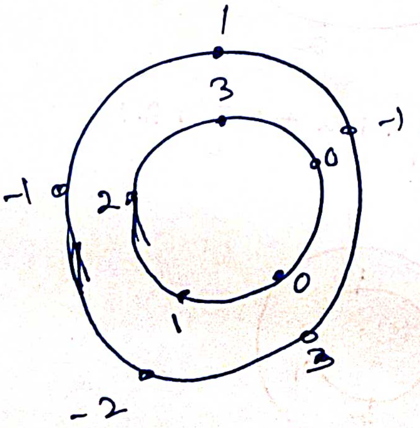


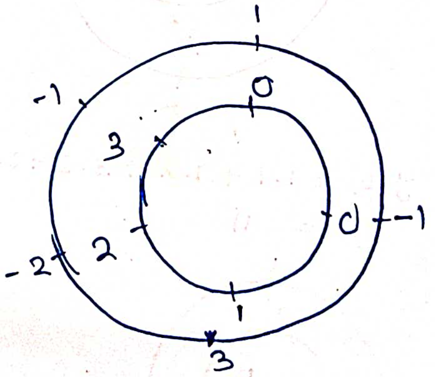
Rotate inner circle 1 bit anticlockwise.



$y(0) = 1.1 + 2(-1) + 3.3 + 0(-2) + 0(-1)$
$= 1 - 2 + 9 + 0 + 0 = 8$

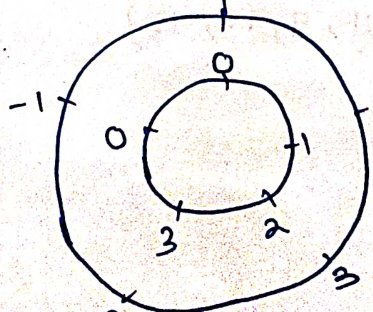
$y(1) = 2(1) + 1(-1) + 0(-2) + 0.3 + 3(-1)$
$= 2 - 1 + 0 + 0 - 3 = -2$





$y(2) = 3(1) + 2(-1) + 1(-2) + 0(3) + 0(-1)$
$= 3 - 2 - 2 + 0 + 0 = -1$

$y(3) = 0(1) + 3(-1) + 2(-2) + 1(3) + 0(-1)$
$= 0 + 3 - 4 + 3 + 0 = -4$



$y(4) = 0(1) + 0(-1) + 3(-2) + 2(3) + 1(-1)$
$= 0 + 0 - 6 + 6 - 1 = -1$

So we get circural convoculion

$y(n) = \{8, -2, -1, -4, -1\}$

The DFT as a Linear Transform:

The formulas for th DFT & IDFT given by

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad K = 0, 1, \cdots N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(K) W_N^{-kn} \qquad n = 0, 1, \cdots N-1$$

where by definition $W_N = e^{-j2\pi/N}$

which is an Nth root of Unity.

- The computation of each point of DFT can be accomplished by N complex multiplication & (N-1) complex additions.
- Hence the N point DFT values can be computed in a total of N^2 complex multiplications & N(N-1) complex addition.

Let us define an N point ~~x~~ vector x_N of the signal sequence $x(n) = 0, 1, \cdots N-1$, an N point vector X_N' of frequency samples & an N X N matrix W_N as

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & & W_N^{(N-1)(N-1)} \end{bmatrix}$$

with these definition, N point DFT may be expressed in matrix as

$$\boxed{X_N = W_N x_N}$$

where W_N is the matrix of the linear Transformation.

Similarly the expression for IDFT is

$$* \quad \boxed{x_N = W_N^{-1} X_N}$$

It is also expressed as

$$\boxed{x_N = \frac{1}{N} W_N^{*} X_N}$$

Where W_N^* denotes the complex conjugate of matrix W_N.

So $W_N^{-1} = \frac{1}{N} W_N^*$

$\Rightarrow W_N . W_N^* =$ ~~NI_N~~ $N I_N$

Where I_N is an $N \times N$ identity Matrix. Therefore, the matrix W_N in the transformation is an orthogonal (unitary) matrix. Its inverse exists & is given as W_N^*/N.

Properties of the DFT:

1) Periodicity:

If $X(k)$ is N point DFT of a finite duration sequence $x(n)$

then $x(n+N) = x(n)$ for all n

$X(K+N) = X(K)$ for all k.

2) Linearity:

If $DFT[x_1(n)] = X_1(K)$

$DFT[x_2(n)] = X_2(K)$

Then $DFT[a x_1(n) + b x_2(n)] = a X_1(K) + b X_2(k)$

3) Time reversal of the Sequence:

If $DFT[x(n)] = X(k)$

Then $DFT[x((-n))_N] = DFT[x(N-n)]$

$= X((-k))_N = X(N-k)$

4) Cincular Time shift :

If $\text{DFT}[x(n)] = X(k)$

then $\text{DFT}[x((n-l))_N] = X(K)\, e^{-j2\pi Kl/N}$

5) Cinculan frequency shift :

If $\text{DFT}[x(n)] = X(k)$

then $\text{DFT}[x(n)\, e^{j2\pi ln/N}] = X((K-l))_N$

6) Complex conjugate property :

If $\text{DFT}[x(n)] = X(k)$

Then $\text{DFT}[x^*(n)] = X^*(N-K) = X^*((-K))_N$

7) Cinculan convolution :

If $\text{DFT}[x_1(n)] = X_1(k)$

$\text{DFT}[x_2(n)] = X_2(k)$

then $\text{DFT}[x_1(n) \textcircled{N} x_2(n)] = X_1(K) . X_2(k)$

⑧ Multiplication of two Sequence :

If $\text{DFT}[x_1(n)] = X_1(k)$

$\text{DFT}[x_2(n)] = X_2(k)$

Then $\text{DFT}[x_1(n) . x_2(n)] = \frac{1}{N} X_1(K) \textcircled{N} X_2(k)$

April
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Relationship of DFT to other transform:

i) Relationship to the fourier transform:

The fourier transform $X(e^{j\omega})$ of a finite duration sequence $x(n)$ having length N is given by

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \qquad -(1)$$

where $X(e^{j\omega})$ is a continuous function of ω.

The discrete fourier transform of $x(n)$ is given by

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi nK}{N}} \qquad K = 0, 1, \cdots N-1 \qquad -(2)$$

comparing equⁿ (1) & (2). we find that the DFT of $x(n)$ is sampled version of fourier transform of sequence.

$$\boxed{X(K) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi K}{N}}}$$

ii) Relationship to the Z transform:

The z transform of a finite sequence $x(n)$ is given by

$$X(Z) = \sum_{n=0}^{N-1} x(n) Z^{-n} \qquad -(1)$$

We know from DFT, $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi nK}{N}}$ $-(2)$

By substituting equⁿ (2) in equⁿ (1) we get

$$X(Z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi nK}{N}} \right] Z^{-n}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \sum_{n=0}^{N-1} \left(e^{\frac{j2\pi K}{N}} \cdot Z^{-1} \right)^{n}$$

$$= \frac{1 - Z^{-N}}{N} \sum_{K=0}^{N-1} \frac{X(K)}{1 - e^{j2\pi K/N} Z^{-1}}$$

②

Properties of Discrete Fourier transform:

(3)

1) Periodicity:

If $X(k)$ is N Point DFT of a finite duration Sequence $x(n)$ then

$$x(n+N) = x(n) \quad \text{for all } n$$

$$X(k+N) = X(k) \quad \text{for all } k.$$

2) Linearity:

If two finite duration sequence $x_1(n)$ & $x_2(n)$ are linearly combined i.e

$$x_3(n) = a x_1(n) + b x_2(n)$$

Then DFT of $x_3(n)$ is

$$X_3(K) = a X_1(k) + b X_2(k).$$

Proof: If $x_1(n)$ has length N_1 and $x_2(n)$ has length N_2 then maximum Length of $x_3(n)$ will be $N_3 = Max(N_1, N_2)$

DFT of $x_1(n)$ is $X_1(K) = \sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi nk}{N}}$

DFT of $x_2(n)$ is $X_2(K) = \sum_{n=0}^{N-1} x_2(n) e^{\frac{-j2\pi nk}{N}}$

if $DFT[x_1(n)] = X_1(K)$

$DFT[x_2(n)] = X_2(k)$

~~Theo~~

Then $\boxed{DFT[a x_1(n) + b x_2(n)] = a X_1(k) + b X_2(k)}$

③

(3) Circular Shift of a sequence:

The periodic extension of the sequence $x(n)$ can be written as

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

$$\Rightarrow x_p(n) = x((n))_N$$

$$= x[(n \text{ modulo } N)]$$

Similarly

$$X_p(k) = X((k))_N$$

(4) Time reversal of the Sequence:

If $DFT[x(n)] = X(k)$

Then $DFT[x((-n))_N] = DFT[x(N-n)]$

$$= X((-k))_N = X(N-k)$$

Proof

$$DFT[x(N-n)] = \sum_{n=0}^{N-1} x(N-n) e^{\frac{-j2\pi nk}{N}}$$

Let by changing index n to $m = N-n$

$$\Rightarrow n = N-m$$

By substituting this

$$DFT[x(N-n)] = \sum_{m=0}^{N-1} x(m) e^{\frac{-j2\pi(N-m)k}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) e^{\frac{-j2\pi \cancel{N}k}{N}} \cdot e^{\frac{j2\pi mk}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi k} \cdot e^{\frac{j2\pi mk}{N}}$$

W_N.

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi mk}{N}}$$

(∵ $e^{-j2\pi K} = 1$ for all value of K)

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi mk}{N}} \cdot 1$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi mk}{N}} \cdot e^{\frac{-j2\pi m N}{N}}$$

(∵ $e^{\frac{-j2\pi m N}{N}} = 1$ for all valu of m)

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi m \frac{(N-K)}{N}}$$

$$= X(N-K)$$

(Proved)

⑤ **Circular frequency shift:**

If DFT $[x(n)] = X(K)$

~~Then DFT $[x(n)\, e^{j2\pi ln/N}$~~ Then DFT $\left[x(n)\, e^{j2\pi ln/N}\right] = X((K-l))_N$

Proof:

$$\text{DFT}\left[x(n)\, e^{j2\pi ln/N}\right] = \sum_{n=0}^{N-1} x(n)\, e^{j2\pi ln/N} \cdot e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(k-l)/N}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(N+k-l)/N}$$

(∵ $e^{-j2\pi n \frac{N}{N}} = 1$ for all n)

$$= X(N+k-l)$$

$$= X((k-l))_N$$

Proved

: The Fast Fourier Transform :

FFT :

- The Fast Fourier Transform (FFT) is an algorithm used to compute of DFT. It makes use of the Symmetry & periodicity properties of twiddle factor W_N^K to effectively reduce the DFT computation time.

- It is based on the fundamental principal of decomposing the computation of DFT of a sequence of length N into successively smaller discrete Fourier Transform

- It provides speed - increase factors.

Direct computation of DFT :

The DFT of a sequence can be evaluated using formula

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \qquad 0 \leq k \leq N-1$$

$$\boxed{W_N = e^{-j2\pi/N}, \text{ which is known as twiddle factor}}$$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \qquad 0 \leq K \leq N-1$$

The direct evaluation of the DFT is basically inefficient because it does not use the Symmetry & periodicity properties of the twiddle factor W_N.

These two properties are :

Symmetry property : $W_N^{K+N/2} = -W_N^K$

Periodicity Property : $W_N^{K+N} = W_N^K$

Types of FFT :

There are basically two classes of FFT algorithms. They are i) Decimation in Time
ii) Decimation in Frequency.

* In decimation - in - time, the sequence for which we need the DFT is successively divided into smaller sequences & the DFTs of these subsequences are combined in a certain pattern to obtain the required DFT of the entire sequence.

* In the decimation - in - frequency approach, the frequency samples of the DFT are decomposed into smaller & smaller subsequences in a similar manner.

i) Decimation - in - time algorithm :

The FFT algorithm is most efficient in calculating N - point DFT. If the number of output points N can be expressed as a power of 2, that is $N = 2^M$, where M is an integer, then this algorithm is known as radix-2 FFT algorithm.

Summary of Step of radix-2 DIT-FFT algorithm:

1. The number of input samples $N = 2^M$, where M is an integer.
   $8 = 2^3$

2. The input sequence is shuffled through bit reversal.

input sample index	Binary Representation	Bit reversed binary	Bit reversal sample index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

3. The number of stages in the flowgraph is given by
   $M = \log_2 N$. , $\underset{\Rightarrow}{Ex}$: $M = 3$

4. Each stage consists of $\frac{N}{2}$ butterflies. $= \frac{8}{2} = 4$.

5. Inputs/Outputs for each butterfly are separated by 2^{m-1} samples, where m represents the stage index, i.e for first stage m=1 & for second stage m=2, so on ?
   & 3rd stage m = 4

6. The number of complex multiplications is given by $\frac{N}{2} \log_2 N$.

Ex:

7. The number of complex additions is given by $N \log_2 N$.

8. The twiddle factor exponents are a function of the stage index m & is given by

$$K = \frac{Nt}{2^m} \quad t = 0, 1, 2 \ldots 2^{m-1} - 1$$

for stage-1 : exponent 0

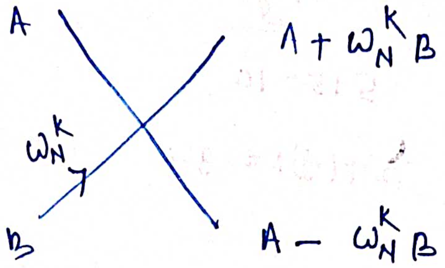
for stage-2 : Exponent 0, 4

for stage 3 : Exponents are 0, 2, 4, 6

9. The number of sets or sections of butterflies in each stage is given by the formula 2^{M-m}

10. The exponent repeat factor (ERF) which is the number of times the exponent sequence associated with m is repeated is given 2^{M-m}.

Flow Graph:



For 8 point twiddle factor.

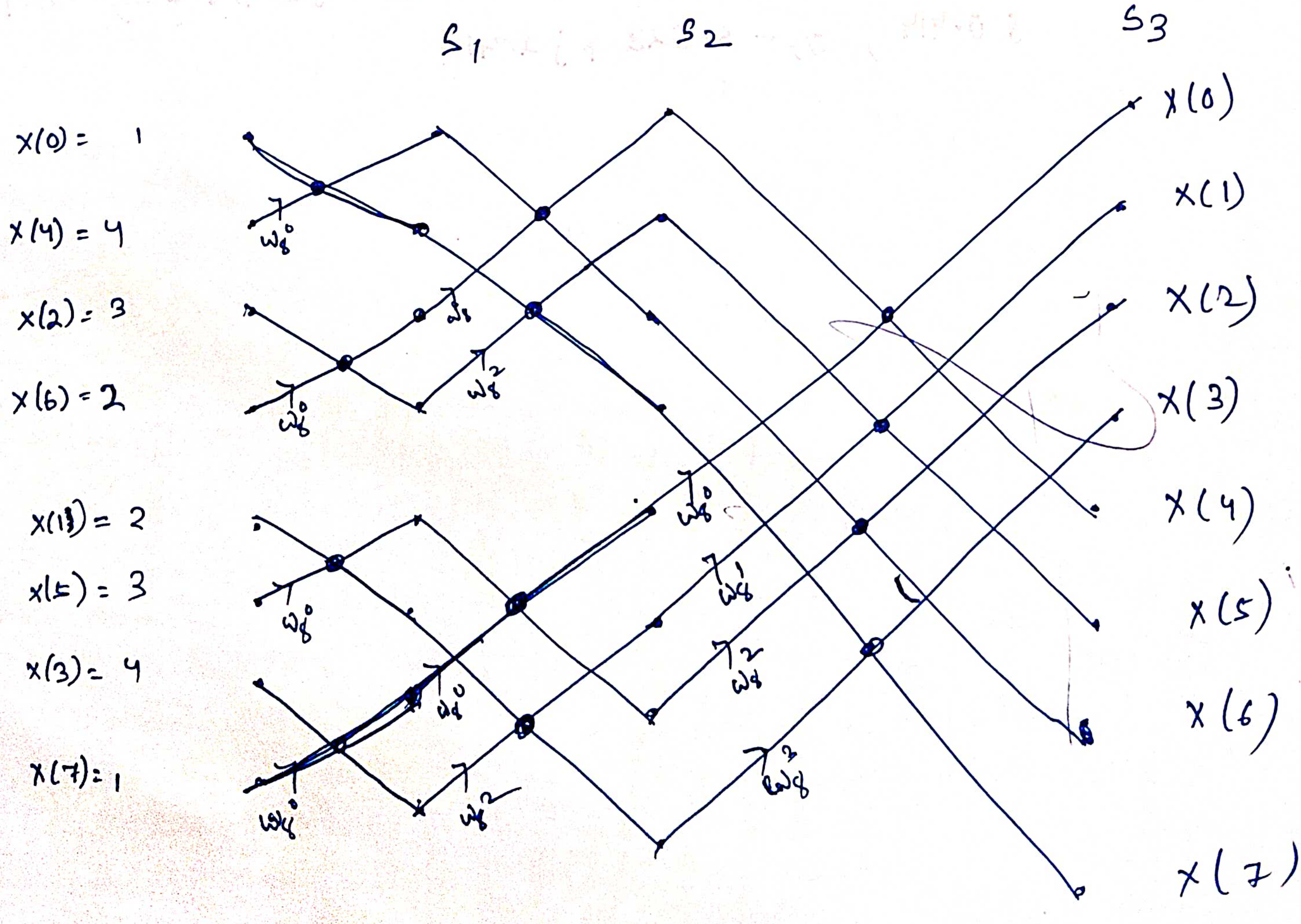
$W_8^0 = 1$, $W_8^1 = e^{\frac{-j2\pi 1}{8}} = e^{-j\pi/4}$ $= \cos\pi/4 - j\sin\pi/4$

$= 0.707 - j0.707$

$W_8^2 = -j$, ~~W_8^3 0.707 − 0.j707~~

$W_8^3 = -0.707 - j0.707$

① Find DFT of a sequence {1, 2, 3, 4, 4, 3, 2, 1} using DIT.



Input	o/p of stage 1	o/p of stage 2	Output
1	$1+4=5$	$5+5=10$	$10+10=20$
4	$1-4=-3$	$-3+(-j)1=-3-j$	$-3-j+(0.707-j0.707)(-1-3j)$ $=-5.828-j2414$
3	$3+2=5$	$5-5=0$	$0+0=0$
2	$3-2=1$	$-3-(-j)1=-3+j$	$(-3+j)+(-.707-j0.707)(-1+3j)$ $=-0.172-j0.414$
2	$2+3=5$	$5+5=10$	$10-10=0$
5	$2-3=-1$	$-1+(-j)3=-1-3j$	$(-3-j)-(0.707-j0.707)(-1-3j)$ $=-0.172+j0.414$
4	$4+1=5$	$5-5=0$	0
1	$4-1=3$	$-1-(-j)3=-1+3j$	$(-3+j)-(-0.707-j0.707)(-1+3j)=-5.828+j2.414$

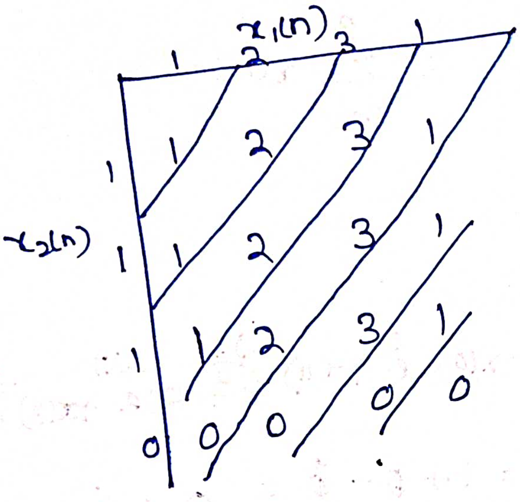
$X(K)=\{20, -5.828-j2.414, 0, -0.172-j0.414, 0, -0.172+j0.414, 0, -5.828+j2.414$

SaiRam

① Find linear convolution & circular convolution.

$x_1(n) = \{1, 2, 3, 1\}$, $x_2(n) = \{1, 1, 1, 0\}$.

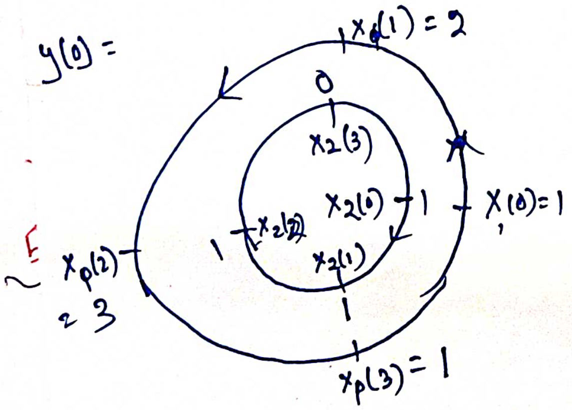
Ans: Linear Convolution.



$y(n) = \{1, 3, 6, 6, 4, 1, 0\}$

Concentric Circle method.

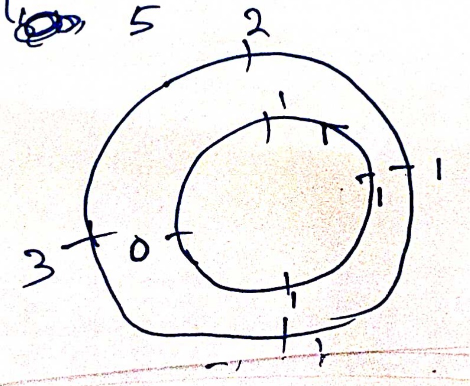
$y(0) =$



$y(0) = \{1.1 + 2.0 + 3.1 + 1.1\}$
$= 5$

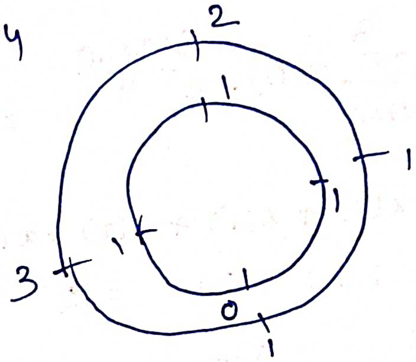
6.

$y(1) =$



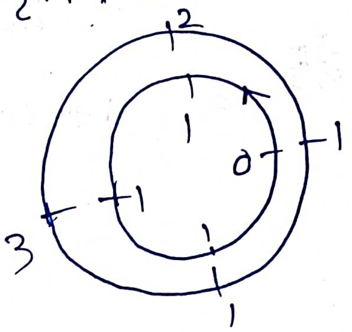
$y(1) = \{1.1 + 2.1 + 3.0 + 1.1\}$
$= 4$

$y(2) =$



$y(2) = \{1.1 + 2.1 + 3.1 + 1.0) = 6$

$y(3) =$



$y(3) = \{1.0 + 2.1 + 3.1 + 1.1\} = 6$

$y(n) = \{5, 4, 6, 6\}$.

Matrix Method

$$\underset{x_2(n)}{\begin{bmatrix}1&0&1&1\\1&1&0&1\\1&1&1&0\\0&1&1&1\end{bmatrix}}\underset{x_1(n)}{\begin{bmatrix}1\\2\\3\\1\end{bmatrix}} = \underset{y(n)}{\begin{bmatrix}1.1+0.2+1.3+1.1\\1.1+1.2+a3+1.1\\1.1+1.2+1.3+a1\\0.1+1.2+1.3+1.1\end{bmatrix}}$$

$$= \underset{y(n)}{\begin{bmatrix}5\\4\\6\\6\end{bmatrix}}$$

$\Rightarrow y(n) = \{5, 4, 6, 6\}$

Saurav

② Find DFT of a Sequence $x(n) = \{1, 1, 0, 0\}$

Ans: Let us assume $L = N = 4$

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}} \qquad \text{where } k = 0, 1, 2 \ldots\ldots N-1$$

$$X(0) = \sum_{n=0}^{3} x(n) \cdot e^{\frac{-j2\pi n \cdot 0}{4}} = \sum_{n=0}^{3} x(n) \cdot e^{0} = \sum_{n=0}^{3} x(n)$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 0 + 0 = 2$$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi n \cdot 1}{4}} = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi n}{2}} = x(0) \cdot e^{0} + x(1)\, e^{\frac{-j\pi}{2}} + x(2)\, e^{-j\pi} + x(3)\, e^{\frac{-j\pi 3}{2}}$$

$$= 1 + 1 \cdot e^{\frac{-j\pi}{2}} = 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = 1 + 0 - j = 1 - j$$

K) =

Define:

(3)

Short Question Answer ①

1) **Zero Padding :** Let the sequence x(n) has a length L. To find N Point DFT (N>L) of the sequence x(n), we have to add (N-L) zero to the sequence x(n). This is known as Zero padding.

2) **Radix 2 FFT :** The FFT algorithm is most efficient calculating N point DFT. If number of output N can be expressed as a power of 2, that is, $N = 2^M$, where M is an integer, then this algorithm is known as "radix 2 FFT algorithm".

3) **FFT :** The ~~fam~~ Fast Fourier transform (FFT) is an algorithm used to compute the DFT. It makes use of symmetry and periodicity properties of twidle factor. It provides speed increase factors.

4) **System function :** The System function of LTI system is defined as the ratio of Y(z) & X(z) i.e

$\boxed{H(z) = \frac{Y(z)}{X(z)}}$ where Y(z) → Z transform of o/p signal y(n)
X(z) → Z transform of i/p signal x(n)

5) **Sampling Theorem :** A band limited contineous time signal, with higher frequency Ωm Hertz, can be uniquely recoved from its samples provided that the sampling rate $\boxed{F \geq 2fm}$ samples per second.

6) **Aliasing effect :**
If the signal x(t) sample with a sampling frequency F < 2fm, the periodic continuation of $X(j\Omega)$ results in spectral overlap. The spectrum $X(j\Omega)$ cannot be recovered using low pass filter. This effect is known as aliasing effect.

7) **Anti-aliasing filter :**
To avoid aliasing, an analog lowpass filter is used before sampler to reshape the frequency spectrum of signal. This filter is known as anti-aliasing filter.

Short

②

8) Quantization:

The process of converting a discrete time continuous amplitude signal x(n) into a discrete - time ~~discrete time~~ determin discrete amplitude signal $x_q(n)$ is known as "quantization".

9. Distinguish between FIR & IIR Filters.

FIR filter	IIR filter
1) The impulse response of this filter ~~has~~ finite number of samples.	1) Impulse response of this filter has an infinite duration.
2) filters have precisely linear phase	2) These filter do not have linear phase.
3) Always stable	3) Not always stable.
4) Greater flexibility to control the shape of their magnitude response.	4) Less flexibility, usually limited to specific kind of filters.
5) These filters can realized recursively and non recursively.	5) IIR filters are easily realized recursively.

10) Design of FIR filters:

There are 3 methods for designing FIR filters.
i) window method ii) Frequency Sampling Method iii) Optimal method

11) What is twiddle factor?

Ans: $W_N = e^{-j2\pi/N}$ is known as twiddle factor.

Symmetry property: $W_N^{k+N/2} = -W_N^k$

Periodicity property: $W_N^{k+N} = W_N^k$

12). The number of multiplication & addition required to compute N point DFT using radix-2 FFT are $N\log_2 N$ and $\frac{N}{2}\log_2 N$ respectively.

Z transform :

Long Question - Answer

③

Determine the Nyquist rate of the below given analog signal.

$m(t) = 2\sin(100\pi t) \cdot \cos(200\pi t) + \cos(50\pi t)$

Ans: $m(t) = 2\sin(100\pi t) \cdot \cos(200\pi t) + \cos(50\pi t)$

$= $ ~~Sin(200+100~~

$= \left[\sin(200\pi t + 100\pi t) - \sin(200\pi t - 100\pi t)\right] + \cos(50\pi t)$

$(\because 2\sin A \cdot \cos B = \sin(A+B) - \sin(A-B))$

$= \sin(300\pi t) - \sin(100\pi t) + \cos(50\pi t)$

Here $f_1 = \frac{300\pi}{2\pi} = 150$ Hz

$f_2 = \frac{100\pi}{2\pi} = 50$ Hz

$f_3 = \frac{50\pi}{2\pi} = 25$ Hz

Here f_{max} maximum frequency of signal = 150Hz

So Nyquist Rate $f_s \geq 2f_{max}$

$f_s = 2 \times 150 = 300$ Hz.

Q ② Find inverse z transform of $X(z) = \dfrac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$ ④

Ans: Multiply both upper & lower by z^2 to eliminate -ve

$$X(z) = \frac{z^2+3z}{z^2+3z+2} = \frac{z(z+3)}{(z+1)(z+2)}$$

Divide $X(z)$ by z

$$\frac{X(z)}{z} = \frac{z+3}{(z+1)(z+2)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{C_1}{z+1} + \frac{C_2}{z+2}$$

$$C_1 = (z+1)\frac{X(z)}{z}\Big|_{z=-1}$$

$$= \cancel{(z+1)}\frac{(z+3)}{\cancel{(z+1)}(z+2)}\Big|_{z=-1} = \frac{(z+3)}{(z+2)}\Big|_{z=-1} = 2$$

$$C_2 = (z+2)\frac{X(z)}{z}\Big|_{z=-2}$$

$$= (z+2)\frac{(z+3)}{(z+1)\cancel{(z+2)}}\Big|_{z=-2} = \frac{(z+3)}{(z+1)}\Big|_{z=-2} = -1$$

Therefore $\dfrac{X(z)}{z} = \dfrac{2}{z+1} - \dfrac{1}{z+2}$

$$\Rightarrow X(z) = 2\frac{z}{z+1} - \frac{z}{z+2}$$

As ROC $|z| > 2$ the sequence is causal & we find

$$x(n) = 2(-1)^n u(n) - (-2)^n u(n)$$

Z transform :

④ Th

⑤

Plot Pole-zero pattern & determine the stability of the system.

$y(n) = 0.7\,y(n-1) - 0.12y\,(n-2) + x(n-1) + x(n-2)$

ns: Taking z transform to both side.

$Y(z) = 0.7\,z^{-1}\,Y(z) - 0.12\,z^{-2}\,Y(z) + z^{-1}X(z) + z^{-2}X(z)$

Now $Y(z) - 0.7\,z^{-1}\,Y(z) + 0.12z^{-2}\,Y(z) = z^{-1}X(z) + z^{-2}X(z)$

$\Rightarrow Y(z)\left[1 - 0.7\,z^{-1} + 0.12z^{-2}\right] = X(z)\left[z^{-1} + z^{-2}\right]$

Now $H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-2}}{1 - 0.7z^{-1} + 0.12z^{-2}}$

$= \cancel{\frac{z^{-1}(1+z^{-1})}{1-0.7z^{-1}+0.12z^{-2}}}$

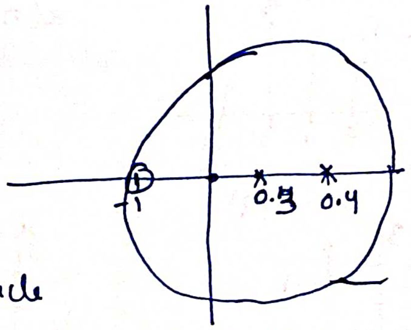
Multipy z^{2} both upper & lower $\Rightarrow H(z) = \frac{z^{2}+1}{z^{2} - 0.7z + 0.12}$

$= \frac{z^{2}+1}{(z-0.4)(z-0.3)}$

Here Pole = $P_1 = 0.4$

$P_2 = 0.3$

zero : $Z_1 = -1$



As both pole & zero lies inside unit circle the system is stable.

[∵ The system is stable if & only if all poles of H(z) are inside the unit circle].

4. Compute poles, zeros & system response of following

$$y(n) = 2y(n-1) + 3x(n)$$

Ans:

$$y(n) = 2y(n-1) + 3x(n)$$

Taking z transform in both side

$$Y(z) = 2z^{-1}Y(z) + 3X(z)$$

$$\Rightarrow Y(z) - 2z^{-1}Y(z) = 3X(z)$$

$$\Rightarrow Y(z)\left[1 - 2z^{-1}\right] = 3X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{3}{1-2z^{-1}}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{3}{1-2z^{-1}}$$

Multiply z in denominator & neumator

$$\boxed{H(z) = \frac{3z}{z-2}}$$

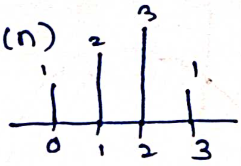
Here Pole, $P_1 = 2$

Zero $Z_1 = 0$

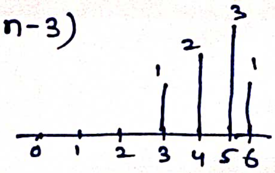
5. Graphically Represent $x(n) = \{1, 2, 3, 1\}$. find $x(2n)$ $x(n/2)$, $x(n+2)$, $x(n-3)$, $x(-n)$ $x(-n+2)$

Ans

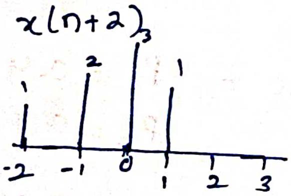
$x(n)$



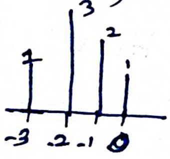
$x(n-3)$



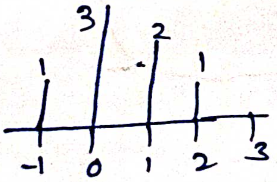
$x(n+2)$



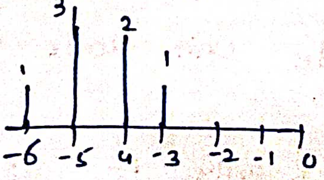
$x(-n)$



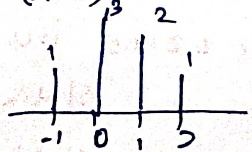
$x(-n+2)$

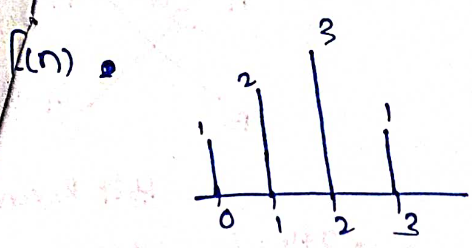


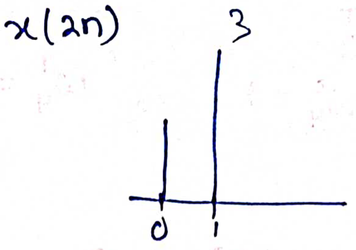
$x(-n-3)$

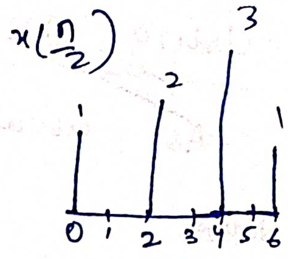


$x(2-n)$









(2n)

$y(0) = x(0) = 1$

$y(1) = x(2) = 3$

$y(2) = x(4) = 0$

x(n/2)

$y(0) = x(0) = 1$

$y(1) = x(\frac{1}{2}) = 0$

$y(2) = x(\frac{2}{2}) = 2$

$y(3) = x(\frac{3}{2})\ 0$

$y(4) = x(\frac{4}{2}) = 3$

$y(5) = x(\frac{5}{2}) = 0$

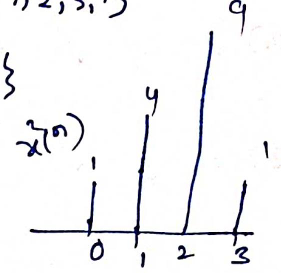
$y(6) \simeq x(\frac{6}{2}) = 1$

$x^2(n)$

$= x(n) . x(n)$

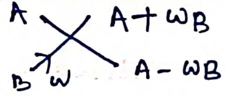
$= \{1,2,3,1\}.\{1,2,3,1\}$

$= \{1, 4, 9, 1\}$

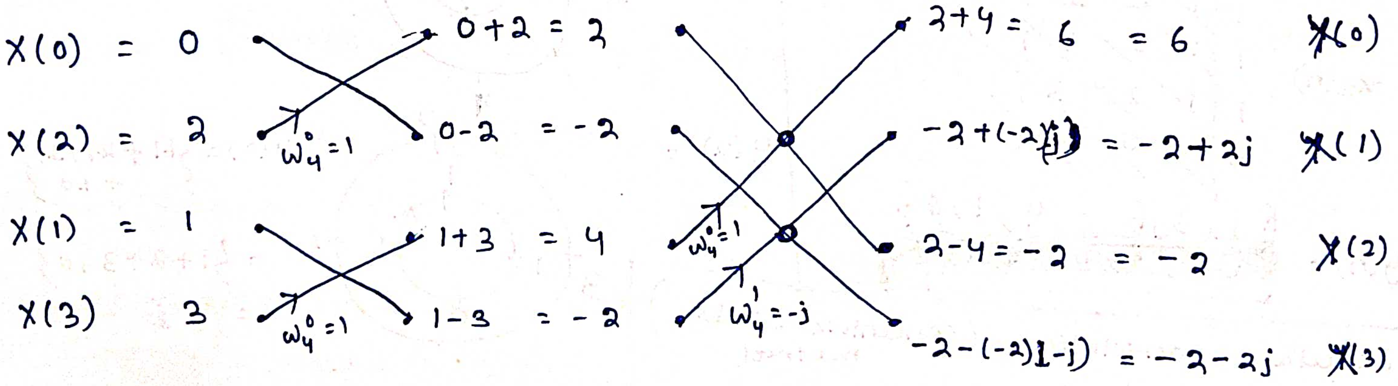


6. Compute 4 Point DFT of a Sequence $x(n) = \{0, 1, 2, 3\}$ using DIT FFT.

Ans: For 4 Point DFT, Twiddle factor is $\boxed{W_4^0 = 1,\ W_4^1 = e^{-j2\pi/4} = -j}$

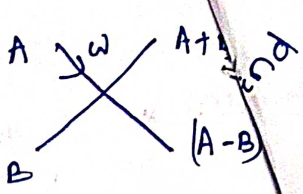


Bit Reversal

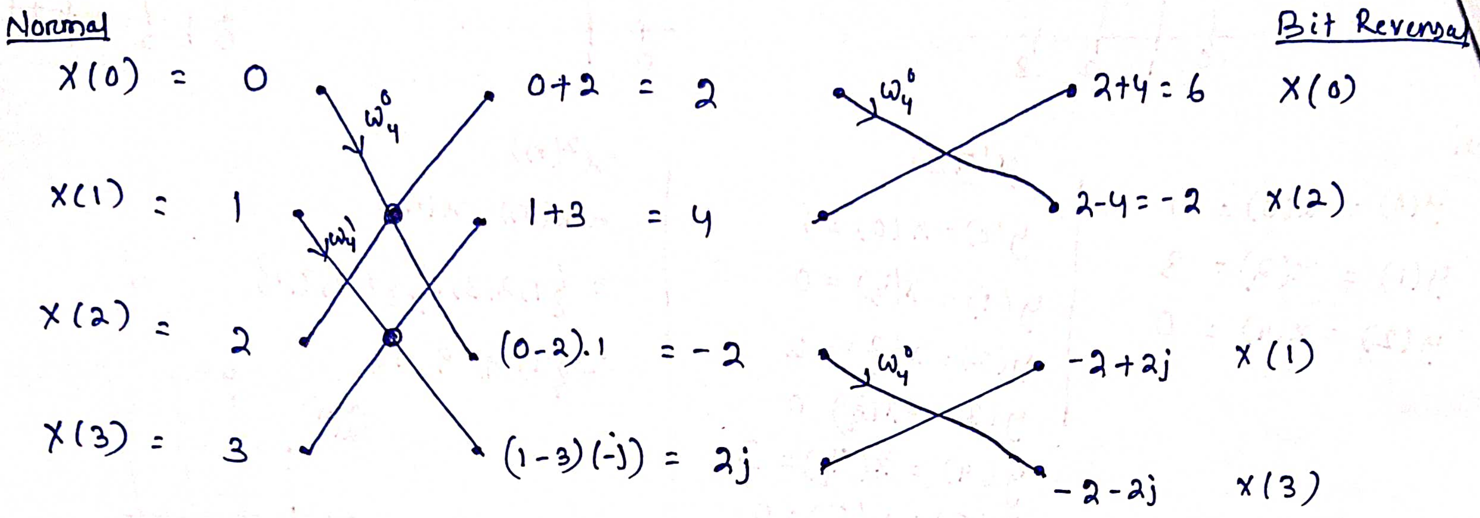


Hence $X(k) = \{6, -2+2j, -2, -2-2j\}$

Using DIF - FFT



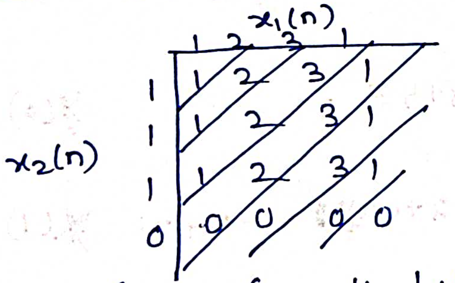
As usual $W_4^0 = 1$, $W_4^1 = -j$



Hence $X(K) = \{6, -2+2j, -2, -2-2j\}$

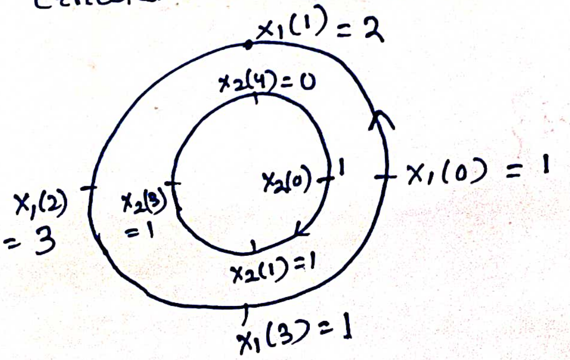
7) Find linear convolution & circular convolution (concentric circle) method. $x_1(n) = \{1, 2, 3, 1\}$, $x_2(n) = \{1, 1, 1, 0\}$

Ans Linear convolution



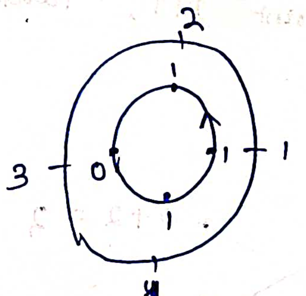
$y(n) = \{1, 3, 6, 6, 4, 1, 0\}$

A Circular convolution (concentric circle method)



$y(0) = \{1.1 + 2.0 + 3.1 + 1.1\} =$
$1 + 0 + 3 + 1 = 5$

$y(1) =$

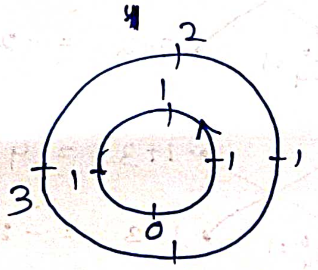


$y(1) = \{1.1 + 2.1 + 3.0 + 1.$[illegible]

$\Rightarrow y(1) = \{1 + 2 + 0 +$ [illegible]$\}$

$= 4$

$y(2) =$

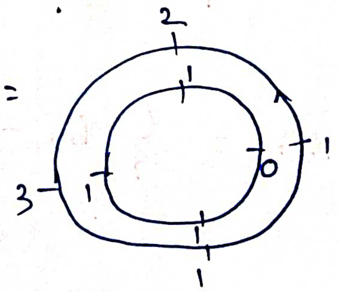


$y(2) = \{1.1 + 2.1 + 3.1 + 1.0\}$

$= \{1 + 2 + 3 + 0\}$

$= 6$

$y(3) =$



$y(3) = \{1.0 + 2.1 + 3.1 + 1.1\}$
$= \{0 + 2 + 3 + 1\} = 6$

$Y(n) = \{5, 4, 6, 6\}$

⑨

nd DFT of a sequence $x(n) = 1$ for $n \leq 0 \leq 2$ for $N = 4$
$= 0$ otherwise.

ns: Given $x(n) = \{1, 1, 1, 0\}$

$L = N = 4$

DFT of a sequence is given by

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \qquad k = 0, 1, \cdots N-1$$

$$X(0) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi n.0}{4}} = \sum_{n=0}^{3} x(n) \cdot 1 \qquad (\because e^{0} = 1)$$

$$= X(0) + X(1) + X(2) + X(3)$$

$$= 1 + 1 + 1 + 0 = 3$$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi n.1}{4}} = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi n}{2}}$$

$$= x(0) \cdot e^{0} + x(1)\, e^{\frac{-j\pi}{2}} + x(2)\, e^{\frac{-j\pi 2}{2}} + x(3)\, e^{\frac{-j\pi 3}{2}} \qquad (\because e^{-j\theta} = \cos\theta - j\sin\theta$$

$$= ~~x(0)e^{0} + x(1)~~ = 1 + 1.\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) + 1.(\cos\pi - j\sin\pi) + 0.$$

$$= 1 + (0 - j.1) + (-1 - j.0) = 1 - j - 1 = -j$$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi n 2}{4}} = \sum_{n=0}^{3} x(n)\, e^{-j\pi n}$$

$$= x(0).1 + x(1)\, e^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}$$

$$= 1 + 1(\cos\pi - j\sin\pi) + 1.\ (\cos 2\pi - j\sin 2\pi) + 0$$

$$= 1 + (-1 - 0) + 1 = 1$$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi n 3}{4}} = \sum_{n=0}^{3} x(n)\, e^{\frac{-j3\pi n}{2}}$$

$$= X(0) + X(1)\, e^{\frac{-j3\pi.1}{2}} + X(2)\, e^{\frac{-j3\pi.2}{2}} + X(3)\, e^{\frac{-j3\pi 3}{2}}$$

$$= 1 + 1.\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 1.\ (\cos 3\pi - j\sin 3\pi) + 0$$

$$= 1 + 1[\,0 - j(-1)] + 1\ (-1 + j.0) = 1 + j - 1 = j$$

Hence $X(K) = \{3, -j, 1, j\}$